AMERICAN MATHEMATICS COMPETITIONS

2nd Annual Mathematics Contest 10

AMC 10

Solutions Pamphlet

TUESDAY, FEBRUARY 13, 2001

Sponsored by

Mathematical Association of America

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This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC "10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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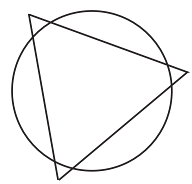
1. **(E)** The middle number in the 9-number list is $n+6$, which is given as 10. Thus $n = 4$. Add the terms together to get $9n + 63 = 9 \cdot 4 + 63 = 99$. Thus the mean is $99/9 = 11$.

2. **(C)** The reciprocal of x is $\frac{1}{x}$, and the additive inverse of x is $-x$. The product of these is $(\frac{1}{x}) \cdot (-x) = -1$. So $x = -1 + 2 = 1$, which is in the interval $0 < x \le 2$.

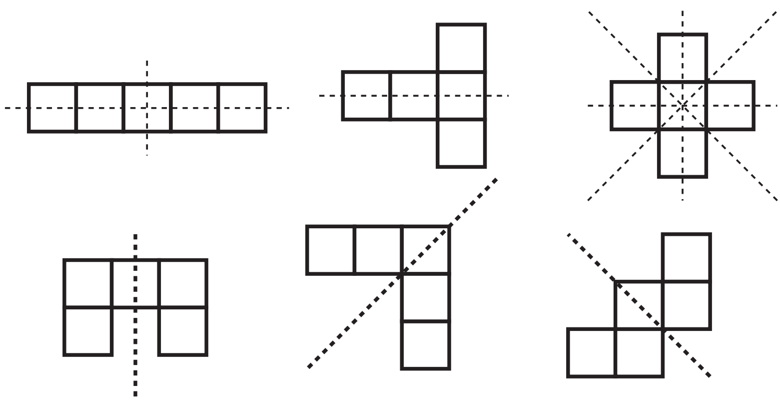
3. **(E)** Suppose the two numbers are a and b. Then the desired sum is

$$2(a_3) + 2(b + 3) = 2(a + b) + 12 = 2S + 12.$$

4. **(E)** The circle can intersect at most two points of each side of the triangle, so the number can be no greater than six. The figure shows that the number can indeed be six.



5. **(D)** Exactly six have at least one line of symmetry. They are:



6. **(E)** Suppose $N = 10a + b$. Then $10a + b = ab + (a + b)$. It follows that $9a = ab$, which implies that $b = 9$, since $a \neq 0$.

7. **(C)** If x is the number, then moving the decimal point four places to the right is the same as multiplying x by 10,000. That is, $10{,}000x = 4 \cdot (\frac{1}{x})$, which is equivalent to $x^2 = 4/10{,}000$. Since x is positive, it follows that $x = 2/100 = 0.02$.

8. **(B)** The number of school days until they will next be together is the least common multiple of 3, 4, 6, and 7, which is 84.

9. **(B)** If Kristin's annual income is $x \geq 28{,}000$ dollars, then

$$\frac{p}{100} \cdot 28{,}000 + \frac{p+2}{100} \cdot (x - 28{,}000) = \frac{p+0.25}{100} \cdot x.$$

Multiplying by 100 and expanding yields

$$28{,}000p + px + 2x - 28{,}000p - 56{,}000 = px + 0.25x.$$

So, $1.75x = \frac{7}{4}x = 56{,}000$ and $x = 32{,}000$.

10. **(D)** Since

$$x = \frac{24}{y} = 48z$$

we have $z = 2y$. So $72 = 2y^2$, which implies that $y = 6$, $x = 4$, and $z = 12$. Hence $x + y + z = 22$.

OR

Take the product of the equations to get $xy \cdot xz \cdot yz = 24 \cdot 48 \cdot 72$. Thus

$$(xyz)^2 = 2^3 \cdot 3 \cdot 2^4 \cdot 3 \cdot 2^3 \cdot 3^2 = 2^{10} \cdot 3^4.$$

So $(xyz)^2 = (2^5 \cdot 3^2)^2$, and we have $xyz = 2^5 \cdot 3^2$. Therefore,

$$x = \frac{xyz}{yz} = \frac{2^5 \cdot 3^2}{2^3 \cdot 3^2} = 4.$$

From this it follows that $y = 6$ and $z = 12$, so the sum is $4 + 6 + 12 = 22$.

11. **(C)** The n^{th} ring can be partitioned into four rectangles: two containing $2n+1$ unit squares and two containing $2n-1$ unit squares. So there are

$$2(2n+1) + 2(2n-1) = 8n$$

unit squares in the n^{th} ring. Thus, the 100^{th} ring has $8 \cdot 100 = 800$ unit squares.

OR

The n^{th} ring can be obtained by removing a square of side $2n-1$ from a square of side $2n+1$. So it contains

$$(2n+1)^2 - (2n-1)^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n$$

unit squares.

12. **(D)** In any triple of consecutive integers, at least one is even and one is a multiple of 3. Therefore, the product of the three integers is both even and a multiple of 3. Since 7 is adivisor of the product, the numbers 6, 14, 21, and 42 must also be divisors of the product. However, 28 contains two factors of 2, and n need not. For example, $5 \cdot 6 \cdot 7$ is divisible by 7, but not by 28.

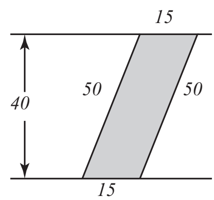
13. **(E)** The last four digits (GHIJ) are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A + B + C = 9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The three digits in DEF are 864, 642, or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such telephone number is 810-642-9753.

14. **(A)** Let n be the number of full-price tickets and p be the price of each in dollars. Then

$$np + (140 - n) \cdot \frac{p}{2} = 2001, \text{ so } p(n + 140) = 4002.$$

Thus $n + 140$ must be a factor of $4002 = 2 \cdot 3 \cdot 23 \cdot 29$. Since $0 \le n \le 140$, we have $140 \le n + 140 \le 280$, and the only factor of 4002 that is in the required range for $n + 140$ is $174 = 2 \cdot 3 \cdot 29$. Therefore, $n + 140 = 174$, so $n = 34$ and $p = 23$. The money raised by the full-price tickets is $34 \cdot 23 = 782$ dollars.

15. **(C)** The crosswalk is in the shape of a parallelogram with base 15 feet and altitude 40 feet, so its area is $15 \times 40 = 600$ ft^2. But viewed another way, the parallelogram has base 50 feet and altitude equal to the distance between the stripes, so this distance must be $600/50 = 12$ feet.



16. **(D)** Since the median is 5, we can write the three numbers as x, 5, and y, where

$$\frac{1}{3}(x + 5 + y) = x + 10 \text{ and } \frac{1}{3}(x + 5 + y) + 15 = y.$$

If we add these equations, we get

$$\frac{2}{3}(x + 5 + y) + 15 = x + y + 10$$

and solving for $x+y$ gives $x+y = 25$. Hence the sum of the numbers $x+y+5 = 30$.

OR

Let m be the mean of the three numbers. Then the least of the numbers is $m - 10$ and the greatest is $m + 15$. The middle of the three numbers is the median, 5. So

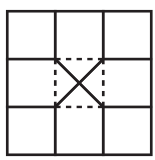
$$\frac{1}{3}((m - 10) + 5 + (m + 15)) = m$$

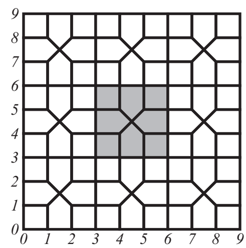
and $m = 10$. Hence, the sum of the three numbers is $3(10) = 30$.

17. **(C)** The slant height of the cone is 10, the radius of the sector. The circmference of the base of the cone is the same as the length of the secotr's arc. This is $252/360 = 7/10$ of the circumference, 20π, of the circle from which the sector is cut. The base circumference of the cone is 14π, so its radius is 7.

18. **(D)** The pattern shown at left is repeated in the plane. In fact, nine repetitions of it are shown in the statement of the problem. Note that four of the nine squres in the three-by-three square are not in the four pentagons that make up the three-by-three square. Therefore, the percentage of the plane that is enclosed by pentagons is

$$1 - \frac{4}{9} = \frac{5}{9} = 55\frac{5}{9}\%$$





19. **(D)** The number of possible selections is the number of solutions of the equation

$$g + c + p = 4$$

where g,c, and p represent, respectively, the number of glazed, chocolate, and powdered donuts. The 15 possible solutions to this equations are (4,0,0), (0,4,0), (0,0,4), (3,0,1),(3,1,0),(1,3,0),(0,3,1),(1,0,3),(0,1,3),(2,2,0),(2,0,2), (0,2,2), (2,1,1),(1,2 and (1,1,2).

OR

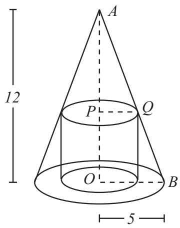
Code each selection as a sequence of four *'s and two —'s, where * represents a donut and each — denotes a "separator" between types of donuts. For example **—*—* represents two glazed donuts, one chocolate donut, and one powdered donut. From the six slots that can be occupied by a — or a *, we must choose two places for the —'s to determine a selection. Thus, there are $\binom{6}{2} \equiv C_2^6 \equiv 6C2 = 15$ selections.

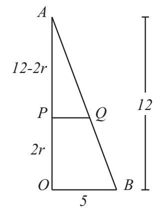
20. **(B)** Let x represent the length of each side of the octagon, which is also the length of the hypotenuse of each of the right triangles. Each leg of the right triangles has length $x\sqrt{2}/2$, so

$$2 \cdot \frac{x\sqrt{2}}{2} + x = 2000, \text{ and } x = \frac{2000}{\sqrt{2}+1} = 2000(\sqrt{2} - 1).$$

21. **(B)** Let the cylinder have radius r and height $2r$. Since $\triangle APQ$ is similar to $\triangle AOB$, we have

$$\frac{12-2r}{r}=\frac{12}{5}, \text{ so } r=\frac{30}{11}$$





22. **(D)** Since v appears in the first row, first column, and on diagonal, the sum of the remaining two numbers in each of these lines must be the same. Thus,

$$25+18=24+w=21+x,$$

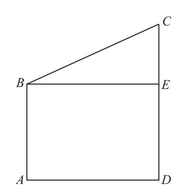
so $w=19$ and $x=22$. now 25,22, and 19 form a diagonal with a sum of 66, so we can find $v=23$, $y=26$, and $z=20$. Hence $y+z=46$.

23. **(D)** Think of continuing the drawing until all five chips are removed form the box. There are ten possible orderings of the colors: RRRWW, RRWRW, RWRRW, WRRRW, RRWWR, RWRWR, WRRWR, RWWRR, WRWRR, and WWRRR. The six orderings that end in R represent drawings that would have ended when the second white chip was drawn.

OR

Imagine drawing until only one chip remains. If the remaining chip is red, then that draw would have ended when the second white chip was removed. The last chip will be red with probability 3/5.

24. **(B)** Let E be the foot of the perpendicular from B to $\overline{CD}$. Then $AB=DE$ and $BE=AD=7$. By the Pythagorean Theorem,



$$\begin{aligned} AD^2=BE^2&=BC^2-CE^2\\ &=(CD+AB)^2-(CD-AB)^2\\ &=(CD+AB+CD-AB)(CD+AB-CD+AB)\\ &=4\cdot CD\cdot AB. \end{aligned}$$

Hence, $AB\cdot CD=AD^2/4=7^2/4=49/4=12.25$.

25. **(B)** For integers not exceeding 2001, there are $\lfloor 2001/3\rfloor=667$ multiples of 3 and $\lfloor 2001/4\rfloor=500$ multiples of 4. The total, 1167, counts the $\lfloor 2001/12\rfloor=166$ multiples of 12 twice, so there are $1167-166=1001$ multiples of 3 or 4. From these we exclude the $\lfloor 2001/15\rfloor=133$ multiples of 15 and the $\lfloor 2001/20\rfloor=100$ multiples of 20, since these are multiples of 5. However, this excludes the $\lfloor 2001/60\rfloor=33$ multiples of 60 twice, so we must re-include these. The number of integers satisfying the conditions is $1001-133-100+33=801$.