The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

3rd Annual American Mathematics Contest 10

AMC 10 - Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 12, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. **(D)** We have

$$\frac{10^{2000}+10^{2002}}{10^{2001}+10^{2001}} = \frac{10^{2000}(1+100)}{10^{2000}(10+10)} = \frac{101}{20} \approx 5.$$

2. **(C)** We have

$$(2,12,9) = \frac{2}{12}+\frac{12}{9}+\frac{9}{2} = \frac{1}{6}+\frac{4}{3}+\frac{9}{2} = \frac{1+8+27}{6} = \frac{36}{6} = 6.$$

3. **(B)** No matter how the exponentiations are performed, 2^{2^2} always gives 16. Depending on which exponentiation is done last, we have

$$\left(2^{2^2}\right)^2 = 256, \quad 2^{\left(2^{2^2}\right)} = 65,536, \quad \text{or} \quad \left(2^2\right)^{\left(2^2\right)} = 256,$$

so there is one other possible value.

4. **(E)** When $n = 1$, the inequality becomes $m \le 1 + m$, which is satisfied by all integers m. Thus, there are infinitely many of the desired values of m.

5. **(C)** The large circle has radius 3, so its area is $\pi \cdot 3^2 = 9\pi$. The seven small circles have a total area of $7\left(\pi \cdot 1^2\right) = 7\pi$. So the shaded region has area $9\pi - 7\pi = 2\pi$.

6. **(A)** Let x be the number she was given. Her calculations produce

$$\frac{x-9}{3} = 43,$$

so

$$x - 9 = 129 \quad \text{and} \quad x = 138.$$

The correct answer is

$$\frac{138-3}{9} = \frac{135}{9} = 15.$$

7. **(A)** Let $C_A = 2\pi R_A$ be the circumference of circle A, let $C_B = 2\pi R_B$ be the circumference of circle B, and let L the common length of the two arcs. Then

$$\frac{45}{360}C_A = L = \frac{30}{360}C_B.$$

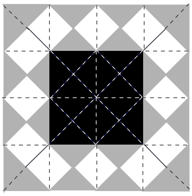
Therefore

$$\frac{C_A}{C_B} = \frac{2}{3} \quad \text{so} \quad \frac{2}{3} = \frac{2\pi R_A}{2\pi R_B} = \frac{R_A}{R_B}.$$

Thus, the ratio of the areas is

$$\frac{\text{Area of Circle }(A)}{\text{Area of Circle }(B)} = \frac{\pi R_A^2}{\pi R_B^2} = \left(\frac{R_A}{R_B}\right)^2 = \frac{4}{9}.$$

8. **(A)** Draw additional lines to cover the entire figure with congruent triangles. There are 24 triangles in the blue region, 24 in the white region, and 16 in the red region. Thus, $B = W$.



9. **(B)** Adding $1001C - 2002A = 4004$ and $1001B + 3003A = 5005$ yields $1001A + 1001B + 1001C = 9009$. So $A + B + C = 9$, and the average is

$$\frac{A + B + C}{3} = 3.$$

10. **(A)** Factor to get $(2x+3)(2x-10) = 0$, so the two roots are $-3/2$ and 5, which sum to $7/2$.

11. **(B)** First note that the amount of memory needed to store the 30 files is

$$3(0.8) + 12(0.7) + 15(0.4) = 16.8 \text{ mb},$$

so the number of disks is at least

$$\frac{16.8}{1.44} = 11 + \frac{2}{3}.$$

However, a disk that contains a 0.8-mb file can, in addition, hold only one 0.4-mb file, so on each of these disks at least 0.24 mb must remain unused. Hence, there is at least $3(0.24) = 0.72$ mb of unused memory, which is equivalent to half a disk. Since

$$\left(11 + \frac{2}{3}\right) + \frac{1}{2} > 12,$$

at least 13 disks are needed.
To see that 13 disks suffice, note that:
Six disks could be used to store the 12 files containing 0.7 mb;
Three disks could be used to store the three 0.8-mb files together with three of the 0.4-mb files;
Four disks could be used to store the remaining twelve 0.4-mb files.

12. **(B)** Let t be the number of hours Mr. Bird must travel to arrive on time. Since three minutes is the same as 0.05 hours, $40(t + 0.05) = 60(t - 0.05)$. Thus,

$$40t + 2 = 60t - 3, \quad \text{so } t = 0.25.$$

The distance from his home to work is $40(0.25 + 0.05) = 12$ miles. Therefore, his average speed should be $12/0.25 = 48$ miles per hour.

OR

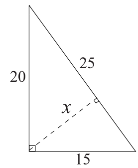
Let d be the distance from Mr. Bird's house to work, and let s be the desired average speed. Then the desired driving time is d/s. Since $d/60$ is three minutes too short and $d/40$ is three minutes too long, the desired time must be the average, so

$$\frac{d}{s} = \frac{1}{2}\left(\frac{d}{60} + \frac{d}{40}\right).$$

This implies that $s = 48$.

13. **(B)** First notice that this is a right triangle, so two of the altitudes are the legs, whose lengths are 15 and 20. The third altitude, whose length is x, is the one drawn to the hypotenuse. The area of the triangle is $\frac{1}{2}(15)(20) = 150$. Using 25 as the base and x as the altitude, we have

$$\frac{1}{2}(25)x = 150, \quad \text{so} \quad x = \frac{300}{25} = 12.$$



OR

Since the three right triangles in the figure are similar,

$$\frac{x}{15} = \frac{20}{25}, \quad \text{so} \quad x = \frac{300}{25} = 12.$$

14. **(B)** Let p and q be two primes that are roots of $x^2 - 63x + k = 0$. Then

$$x^2 - 63x + k = (x-p)(x-q) = x^2 - (p+q)x + p \cdot q,$$

so $p + q = 63$ and $p \cdot q = k$. Since 63 is odd, one of the primes must be 2 and the other 61. Thus, there is exactly one possible value for k, namely $k = p \cdot q = 2 \cdot 61 = 122$.

15. **(E)** The digits 2, 4, 5, and 6 cannot be the units digit of any two-digit prime, so these four digits must be the tens digits, and 1, 3, 7, and 9 are the units digits. The sum is thus

$$10(2+4+5+6) + (1+3+7+9) = 190.$$

(One set that satisfies the conditions is $\{23, 47, 59, 61\}$.)

16. **(B)** From the given information,

$$(a+1) + (b+2) + (c+3) + (d+4) = 4(a+b+c+d+5),$$

so

$$(a+b+c+d) + 10 = 4(a+b+c+d) + 20$$

and $a + b + c + d = -\dfrac{10}{3}$.

OR

Note that $a = d + 3$, $b = d + 2$, and $c = d + 1$. So,

$$a + b + c + d = (d+3) + (d+2) + (d+1) + d = 4d + 6.$$

Thus, $d + 4 = (4d + 6) + 5$, so $d = -7/3$, and

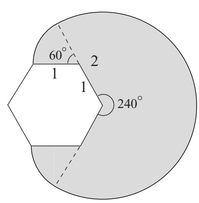
$$a + b + c + d = 4d + 6 = 4\left(-\frac{7}{3}\right) + 6 = -\frac{10}{3}.$$

17. **(D)** After the first transfer, the first cup contains two ounces of coffee, and the second cup contains two ounces of coffee and four ounces of cream. After the second transfer, the first cup contains $2 + (1/2)(2) = 3$ ounces of coffee and $(1/2)(4) = 2$ ounces of cream. Therefore, the fraction of the liquid in the first cup that is cream is $2/(2+3) = 2/5$.

18. **(D)** There are six dice that have a single face on the surface, and these dice can be oriented so that the face with the 1 is showing. They will contribute $6(1) = 6$ to the sum. There are twelve dice that have just two faces on the surface because they are along an edge but not at a vertex of the large cube. These dice can be oriented so that the 1 and 2 are showing, and they will contribute $12(1+2) = 36$ to the sum. There are eight dice that have three faces on the surface because they are at the vertices of the large cube, and these dice can be oriented so that the 1, 2, and 3 are showing. They will contribute $8(1+2+3) = 48$ to the sum. Consequently, the minimum sum of all the numbers showing on the large cube is $6 + 36 + 48 = 90$.

19. **(E)** Spot can go anywhere in a $240°$ sector of radius two yards and can cover a $60°$ sector of radius one yard around each of the adjoining corners. The total area is

$$\pi(2)^2 \cdot \frac{240}{360} + 2\left(\pi(1)^2 \cdot \frac{60}{360}\right) = 3\pi.$$



20. **(D)** Since $\triangle AGD$ is similar to $\triangle CHD$, we have $HC/1 = AG/3$. Also, $\triangle AGF$ is similar to $\triangle EJF$, so $JE/1 = AG/5$. Hence,

$$\frac{HC}{JE} = \frac{AG/3}{AG/5} = \frac{5}{3}.$$

21. **(D)** The values 6, 6, 6, 8, 8, 8, 8, 14 satisfy the requirements of the problem, so the answer is at least 14. If the largest number were 15, the collection would have the ordered form 7, __, __, 8, 8, __, __, 15. But $7 + 8 + 8 + 15 = 38$, and a mean of 8 implies that the sum of all values is 64. In this case, the four missing values would sum to $64 - 38 = 26$, and their average value would be 6.5. This implies that at least one would be less than 7, which is a contradiction. Therefore, the largest integer that can be in the set is 14.

22. **(C)** The first application removes ten tiles, leaving 90. The second and third applications each remove nine tiles leaving 81 and 72, respectively. Following this pattern, we consecutively remove $10, 9, 9, 8, 8, \ldots, 2, 2, 1$ tiles before we are left with only one. This requires $1 + 2(8) + 1 = 18$ applications.

OR

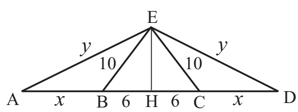
Starting with n^2 tiles, the first application leaves $n^2 - n$ tiles. The second application reduces the number to $n^2 - n - (n-1) = (n-1)^2$ tiles. Since two applications reduce the number from n^2 to $(n-1)^2$, it follows that $2(n-1)$ applications reduce the number from n^2 to $(n-(n-1))^2 = 1$, and $2(10-1) = 18$.

23. **(D)** Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles. Segment $\overline{EH}$ is the common altitude of the two isosceles triangles $\triangle AED$ and $\triangle BEC$, and

$$EH = \sqrt{10^2 - 6^2} = 8.$$

Let $AB = CD = x$ and $AE = ED = y$. Then $2x+2y+12 = 2(32)$, so $y = 26-x$. Thus,

$$8^2 + (x+6)^2 = y^2 = (26-x)^2 \quad \text{and} \quad x = 9.$$

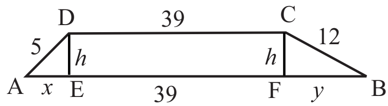


24. **(A)** There are ten ways for Tina to select a pair of numbers. The sums 9, 8, 4, and 3 can be obtained in just one way, and the sums 7, 6, and 5 can each be obtained in two ways. The probability for each of Sergio's choices is 1/10. Considering his selections in decreasing order, the total probability of Sergio's choice being greater is

$$\left(\frac{1}{10}\right)\left(1 + \frac{9}{10} + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} + 0 + 0 + 0\right) = \frac{2}{5}.$$

25. **(C)** First drop perpendiculars from D and C to $\overline{AB}$. Let E and F be the feet of the perpendiculars to $\overline{AB}$ from D and C, respectively, and let

$$h = DE = CF, \quad x = AE, \quad \text{and} \quad y = FB.$$



Then

$$25 = h^2 + x^2, \quad 144 = h^2 + y^2, \quad \text{and} \quad 13 = x + y.$$

So

$$144 = h^2 + y^2 = h^2 + (13 - x)^2 = h^2 + x^2 + 169 - 26x = 25 + 169 - 26x,$$

which gives $x = 50/26 = 25/13$, and

$$h = \sqrt{5^2 - \left(\frac{25}{13}\right)^2} = 5\sqrt{1 - \frac{25}{169}} = 5\sqrt{\frac{144}{169}} = \frac{60}{13}.$$

Hence

$$\text{Area } (ABCD) = \frac{1}{2}(39 + 52) \cdot \frac{60}{13} = 210.$$

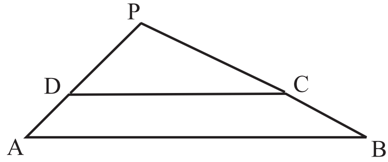
OR

Extend $\overline{AD}$ and $\overline{BC}$ to intersect at P. Since $\triangle PDC$ and $\triangle PAB$ are similar, we have

$$\frac{PD}{PD + 5} = \frac{39}{52} = \frac{PC}{PC + 12}.$$

So $PD = 15$ and $PC = 36$. Note that 15, 36, and 39 are three times 5, 12, and 13, respectively, so $\angle APB$ is a right angle. The area of the trapezoid is the difference of the areas of $\triangle PAB$ and $\triangle PDC$, so

$$\text{Area}(ABCD) = \frac{1}{2}(20)(48) - \frac{1}{2}(15)(36) = 210.$$



OR

Draw the line through D parallel to $\overline{BC}$, intersecting $\overline{AB}$ at E. Then $BCDE$ is a parallelogram, so $DE = 12$, $EB = 39$, and $AE = 52 - 39 = 13$. Thus $DE^2 + AD^2 = AE^2$, and $\triangle ADE$ is a right triangle. Let h be the altitude from D to $\overline{AE}$, and note that

$$\text{Area}(ADE) = \frac{1}{2}(5)(12) = \frac{1}{2}(13)(h),$$

so $h = 60/13$. Thus

$$\text{Area}(ABCD) = \frac{60}{13} \cdot \frac{1}{2}(39 + 52) = 210.$$

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