The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

3rd Annual American Mathematics Contest 10

AMC 10 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 27, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. **(E)** We have

$$\frac{2^{2001}\cdot 3^{2003}}{6^{2002}} = \frac{2^{2001}\cdot 3^{2003}}{(2\cdot 3)^{2002}} = \frac{2^{2001}\cdot 3^{2003}}{2^{2002}\cdot 3^{2002}} = \frac{3}{2}.$$

2. **(C)** We have

$$(2,4,6) = \frac{2\cdot 4\cdot 6}{2+4+6} = \frac{48}{12} = 4.$$

3. **(A)** The number M is equal to

$$\frac{1}{9}(9+99+999+\ldots+999{,}999{,}999) = 1+11+111+\ldots+111{,}111{,}111 = 123{,}456{,}789.$$

The number M does not contain the digit 0.

4. **(D)** Since

$$\begin{aligned}(3x-2)(4x+1)-(3x-2)4x+1 &= (3x-2)(4x+1-4x)+1\\ &= (3x-2)\cdot 1+1 = 3x-1,\end{aligned}$$

when $x = 4$ we have the value $3\cdot 4 - 1 = 11$.

5. **(E)** The diameter of the large circle is $6+4 = 10$, so its radius is 5. Hence, the area of the shaded region is

$$\pi(5^2) - \pi(3^2) - \pi(2^2) = \pi(25-9-4) = 12\pi.$$

6. **(B)** If $n \geq 4$, then

$$n^2 - 3n + 2 = (n-1)(n-2)$$

is the product of two integers greater than 1, and thus is not prime. For $n = 1$, 2, and 3 we have, respectively,

$$(1-1)(1-2) = 0, \quad (2-1)(2-2) = 0, \quad \text{and} \quad (3-1)(3-2) = 2.$$

Therefore, $n^2 - 3n + 2$ is prime only when $n = 3$.

7. **(E)** The number $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is greater than 0 and less than $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{1}<2$. Hence,
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}=\frac{41}{42}+\frac{1}{n}$$
is an integer precisely when it is equal to 1. This implies that $n=42$, so the answer is (E).

8. **(D)** Since July has 31 days, Monday must be one of the last three days of July. Therefore, Thursday must be one of the first three days of August, which also has 31 days. So Thursday must occur five times in August.

9. **(D)** The last "word," which occupies position 120, is *USOMA*. Immediately preceding this we have *USOAM*, *USMOA*, *USMAO*, *USAOM*,and *USAMO*. The alphabetic position of the word *USAMO* is consequently 115.

10. **(C)** The given conditions imply that
$$x^2+ax+b=(x-a)(x-b)=x^2-(a+b)x+ab,$$
so
$$a+b=-a \quad \text{and} \quad ab=b.$$
Since $b\neq 0$, the second equation implies that $a=1$. The first equation gives $b=-2$, so $(a,b)=(1,-2)$.

11. **(B)** Let $n-1$, n, and $n+1$ denote the three integers. Then
$$(n-1)n(n+1)=8(3n).$$
Since $n\neq 0$, we have $n^2-1=24$. It follows that $n^2=25$ and $n=5$. Thus,
$$(n-1)^2+n^2+(n+1)^2=16+25+36=77.$$

12. **(E)** From the given equation we have $(x-1)(x-6) = (x-2)(x-k)$. This implies that

$$x^2 - 7x + 6 = x^2 - (2+k)x + 2k,$$

so

$$(k-5)x = 2k-6 \quad \text{and} \quad x = \frac{2k-6}{k-5}.$$

Hence a value of x satisfying the equation occurs unless $k = 5$.
Note that when $k = 6$ there is also no solution for x, but this is not one of the answer choices.

13. **(D)** The given equation can be factored as

$$0 = 8xy - 12y + 2x - 3 = 4y(2x-3) + (2x-3) = (4y+1)(2x-3).$$

For this equation to be true for all values of y we must have $2x - 3 = 0$, that is, $x = 3/2$.

14. **(B)** We have

$$N = \sqrt{(5^2)^{64} \cdot (2^6)^{25}} = 5^{64} \cdot 2^{3 \cdot 25} = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048 = 2048\underbrace{000\cdots0}_{64 \text{ digits}}.$$

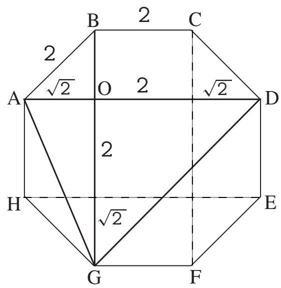
The zeros do not contribute to the sum, so the sum of the digits of N is $2+4+8 = 14$.

15. **(E)** The numbers $A-B$ and $A+B$ are both odd or both even. However, they are also both prime, so they must both be odd. Therefore, one of A and B is odd and the other even. Because A is a prime between $A-B$ and $A+B$, A must be the odd prime. Therefore, $B = 2$, the only even prime. So $A-2$, A, and $A+2$ are consecutive odd primes and thus must be 3, 5, and 7. The sum of the four primes 2, 3, 5, and 7 is the prime number 17.

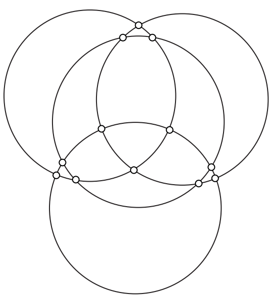
16. **(D)** If $\frac{n}{20-n} = k^2$, for some $k \geq 0$, then $n = \frac{20k^2}{k^2+1}$. Since k^2 and k^2+1 have no common factors and n is an integer, k^2+1 must be a factor of 20. This occurs only when $k = 0, 1, 2,$ or 3. The corresponding values of n are $0, 10, 16,$ and 18.

17. **(C)** Construct the right triangle $\triangle AOB$ as shown in the figure. Since $AB = 2$, we have $AO = \sqrt{2}$ and $AD = 2 + 2\sqrt{2}$. Similarly, we have $OG = 2 + \sqrt{2}$, so

$$\text{Area}(\triangle ADG) = \frac{1}{2}(2 + 2\sqrt{2})(2 + \sqrt{2}) = (1 + \sqrt{2})(2 + \sqrt{2}) = 4 + 3\sqrt{2}.$$



18. **(D)** Each pair of circles has at most two intersection points. There are $\binom{4}{2} = 6$ pairs of circles, so there are at most $6 \times 2 = 12$ points of intersection. The following configuration shows that 12 points of intersection are indeed possible:



19. **(C)** Let $d = a_2 - a_1$. Then $a_{k+100} = a_k + 100d$, and

$$\begin{aligned} a_{101} + a_{102} + \cdots + a_{200} &= (a_1 + 100d) + (a_2 + 100d) + \ldots + (a_{100} + 100d) \\ &= a_1 + a_2 + \ldots + a_{100} + 10{,}000d. \end{aligned}$$

Thus $200 = 100 + 10{,}000d$ and $d = \frac{100}{10{,}000} = 0.01$.

20. **(B)** We have $a+8c=4+7b$ and $8a-c=7-4b$. Squaring both equations and adding the results yields

$$(a+8c)^2+(8a-c)^2=(4+7b)^2+(7-4b)^2.$$

Expanding gives $65(a^2+c^2)=65(1+b^2)$. So $a^2+c^2=1+b^2$, and $a^2-b^2+c^2=1$.

21. **(B)** Let A be the number of square feet in Andy's lawn. Then $A/2$ and $A/3$ are the areas of Beth's lawn and Carlos' lawn, respectively, in square feet. Let R be the rate, in square feet per minute, that Carlos' lawn mower cuts. Then Beth's mower and Andy's mower cut at rates of $2R$ and $3R$ square feet per minute, respectively. Thus,

$$\text{Andy takes } \frac{A}{3R} \text{ minutes to mow his lawn,}$$

$$\text{Beth takes } \frac{A/2}{2R}=\frac{A}{4R} \text{ minutes to mow hers,}$$

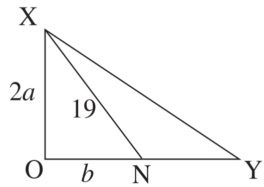
and

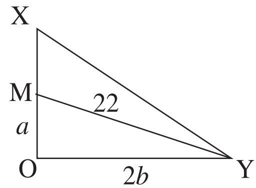
$$\text{Carlos takes } \frac{A/3}{R}=\frac{A}{3R} \text{ minutes to mow his.}$$

Since $\dfrac{A}{4R}<\dfrac{A}{3R}$, Beth will finish first.

22. **(B)** Let $OM=a$ and $ON=b$. Then

$$19^2=(2a)^2+b^2 \quad \text{and} \quad 22^2=a^2+(2b)^2.$$





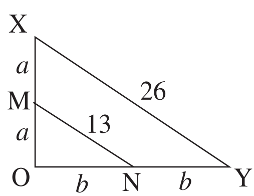
Hence

$$5(a^2+b^2)=19^2+22^2=845.$$

It follows that

$$MN=\sqrt{a^2+b^2}=\sqrt{169}=13.$$

Since $\triangle XOY$ is similar to $\triangle MON$ and $XO=2\cdot MO$, we have $XY=2\cdot MN=26$.



23. **(D)** By setting $n = 1$ in the given recursive equation, we obtain $a_{m+1} = a_m + a_1 + m$, for all positive integers m. So $a_{m+1} - a_m = m+1$ for each $m = 1, 2, 3, \ldots$. Hence,

$$a_{12} - a_{11} = 12,\ a_{11} - a_{10} = 11,\ \ldots,\ a_2 - a_1 = 2.$$

Summing these equalities yields $a_{12} - a_1 = 12 + 11 + \cdots + 2$. So

$$a_{12} = 12 + 11 + \cdots + 2 + 1 = \frac{12(12+1)}{2} = 78.$$

OR

We have

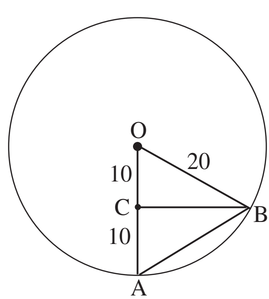
$$\begin{aligned} a_2 &= a_{1+1} = a_1 + a_1 + 1 \cdot 1 = 1 + 1 + 1 = 3, \\ a_3 &= a_{2+1} = a_2 + a_1 + 2 \cdot 1 = 3 + 1 + 2 = 6, \\ a_6 &= a_{3+3} = a_3 + a_3 + 3 \cdot 3 = 6 + 6 + 9 = 21, \end{aligned}$$

and

$$a_{12} = a_{6+6} = a_6 + a_6 + 6 \cdot 6 = 21 + 21 + 36 = 78.$$

24. **(D)** In the figure, the center of the wheel is at O, and the rider travels from A to B. Since $AC = 10$ and $OB = OA = 20$, the point C is the midpoint of $\overline{OA}$. In the right $\triangle OCB$, we have OC half of the length of the hypotenuse OB, so $m\angle COB = 60^\circ$. Since the wheel turns through an angle of 360° in 60 seconds, the time required to turn through an angle of 60° is

$$60\left(\frac{60}{360}\right) = 10 \text{ seconds.}$$



25. **(A)** Let n denote the number of integers in the original list, and m the original mean. Then the sum of the original numbers is mn. After 15 is appended to the list, we have the sum

$$(m+2)(n+1) = mn + 15, \quad \text{so} \quad m + 2n = 13.$$

After 1 is appended to the enlarged list, we have the sum

$$(m+1)(n+2) = mn + 16, \quad \text{so} \quad 2m + n = 14.$$

Solving $m + 2n = 13$ and $2m + n = 14$ gives $m = 5$ and $n = 4$.

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