The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

4th Annual American Mathematics Contest 10

AMC 10 - Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 11, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. (D) Each even counting number, beginning with 2, is one more than the preceding odd counting number. Therefore the difference is $(1)(2003) = 2003$.

2. (B) The cost for each member is the price of two pairs of socks, \$8, and two shirts, \$18, for a total of \$26. So there are $2366/26 = 91$ members.

3. (D) The total volume of the eight removed cubes is $8 \times 3^3 = 216$ cubic centimeters, and the volume of the original box is $15 \times 10 \times 8 = 1200$ cubic centimeters. Therefore the volume has been reduced by $\left(\frac{216}{1200}\right)(100\%) = 18\%$.

4. (A) Mary walks a total of 2 km in 40 minutes. Because 40 minutes is 2/3 hr, her average speed, in km/hr, is $2/(2/3) = 3$.

5. (B) Since

$$0 = 2x^2 + 3x - 5 = (2x+5)(x-1) \quad \text{we have} \quad d = -\frac{5}{2} \text{ and } e = 1.$$

So $(d-1)(e-1) = 0$.

OR

If $x = d$ and $x = e$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$de = \frac{c}{a} \quad \text{and} \quad d + e = -\frac{b}{a}.$$

For our equation this implies that

$$(d-1)(e-1) = de - (d+e) + 1 = -\frac{5}{2} - \left(-\frac{3}{2}\right) + 1 = 0.$$

6. (C) For example, $-1\heartsuit 0 = |-1-0| = 1 \neq -1$. All the other statements are true:
(*A*) $x\heartsuit y = |x-y| = |-(y-x)| = |y-x| = y\heartsuit x$ for all x and y.
(*B*) $2(x\heartsuit y) = 2|x-y| = |2x-2y| = (2x)\heartsuit(2y)$ for all x and y.
(*D*) $x\heartsuit x = |x-x| = 0$ for all x.
(*E*) $x\heartsuit y = |x-y| > 0$ if $x \neq y$.

7. (B) The longest side cannot be greater than 3, since otherwise the remaining two sides would not be long enough to form a triangle. The only possible triangles have side lengths 1–3–3 or 2–2–3.

8. (E) The factors of 60 are

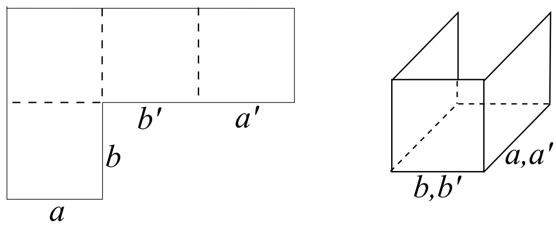
$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, \text{ and } 60.$$

Six of the twelve factors are less than 7, so the probability is 1/2.

9. (A) We have

$$\begin{aligned}\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt{x}}}} &= (x(x(x\cdot x^{\frac{1}{2}})^{\frac{1}{3}})^{\frac{1}{3}})^{\frac{1}{3}} \\ &= (x(x(x^{\frac{3}{2}})^{\frac{1}{3}})^{\frac{1}{3}})^{\frac{1}{3}} \\ &= (x(x\cdot x^{\frac{1}{2}})^{\frac{1}{3}})^{\frac{1}{3}} \\ &= (x(x^{\frac{3}{2}})^{\frac{1}{3}})^{\frac{1}{3}} = (x\cdot x^{\frac{1}{2}})^{\frac{1}{3}} = (x^{\frac{3}{2}})^{\frac{1}{3}} = x^{\frac{1}{2}} = \sqrt{x}.\end{aligned}$$

10. (E) If the polygon is folded before the fifth square is attached, then edges a and a' must be joined, as must b and b'. The fifth face of the cube can be attached at any of the six remaining edges.

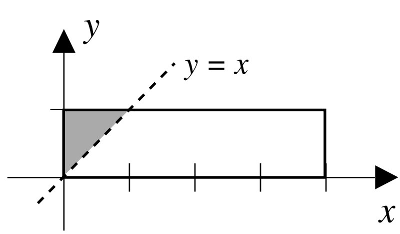


11. (E) Since the last two digits of $AMC10$ and $AMC12$ sum to 22, we have

$$AMC + AMC = 2(AMC) = 1234.$$

Hence $AMC = 617$, so $A = 6$, $M = 1$, $C = 7$, and $A+M+C = 6+1+7 = 14$.

12. (A) The point (x, y) satisfies $x < y$ if and only if it belongs to the shaded triangle bounded by the lines $x = y$, $y = 1$, and $x = 0$, the area of which is $1/2$. The ratio of the area of the triangle to the area of the rectangle is $\frac{1/2}{4} = \frac{1}{8}$.



13. (A) Let a, b, and c be the three numbers. Replace a by four times the sum of the other two to get

$$4(b+c) + b + c = 20, \quad \text{so} \quad b + c = 4.$$

Then replace b with $7c$ to get

$$7c + c = 4, \quad \text{so} \quad c = \frac{1}{2}.$$

The other two numbers are $b = 7/2$ and $a = 16$, and the product of the three is $16 \cdot 7/2 \cdot 1/2 = 28$.

OR

Let the first, second, and third numbers be x, $7x$, and $32x$, respectively. Then $40x = 20$ so $x = \frac{1}{2}$ and the product is

$$(32)(7)x^3 = (32)(7)\left(\frac{1}{8}\right) = 28.$$

14. (A) The largest single-digit primes are 5 and 7, but neither 75 nor 57 is prime. Using 3, 7, and 73 gives 1533, whose digits have a sum of 12.

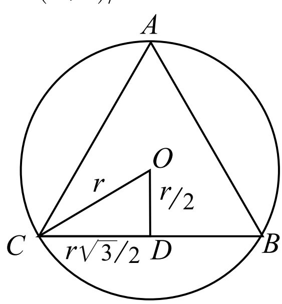
15. (C) Of the $\frac{100}{2} = 50$ integers that are divisible by 2, there are $\lfloor \frac{100}{6} \rfloor = 16$ that are divisible by both 2 and 3. So there are $50 - 16 = 34$ that are divisible by 2 and not by 3, and $34/100 = 17/50$.

16. (C) Powers of 13 have the same units digit as the corresponding powers of 3; and

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \quad \text{and} \quad 3^5 = 243.$$

Since the units digit of 3^1 is the same as the units digit of 3^5, units digits of powers of 3 cycle through $3, 9, 7$, and 1. Hence the units digit of 3^{2000} is 1, so the units digit of 3^{2003} is 7. The same is true of the units digit of 13^{2003}.

17. (B) Let the triangle have vertices A, B, and C, let O be the center of the circle, and let D be the midpoint of $\overline{BC}$. Triangle COD is a 30–60–90 degree triangle. If r is the radius of the circle, then the sides of $\triangle COD$ are r, $r/2$, and $r\sqrt{3}/2$. The perimeter of $\triangle ABC$ is $6\left(\frac{r\sqrt{3}}{2}\right) = 3r\sqrt{3}$, and the area of the circle is πr^2. Thus $3r\sqrt{3} = \pi r^2$, and $r = (3\sqrt{3})/\pi$.



18. (B) Let $a = 2003/2004$. The given equation is equivalent to

$$ax^2 + x + 1 = 0.$$

If the roots of this equation are denoted r and s, then

$$rs = \frac{1}{a} \quad \text{and} \quad r + s = -\frac{1}{a},$$

so

$$\frac{1}{r}+\frac{1}{s}=\frac{r+s}{rs}=-1.$$

OR

If x is replaced by $1/y$, then the roots of the resulting equation are the reciprocals of the roots of the original equation. The new equation is

$$\frac{2003}{2004y}+1+y=0 \quad \text{which is equivalent to} \quad y^2+y+\frac{2003}{2004}=0.$$

The sum of the roots of this equation is the opposite of the y-coefficient, which is -1.

19. (C) First note that the area of the region determined by the triangle topped by the semicircle of diameter 1 is

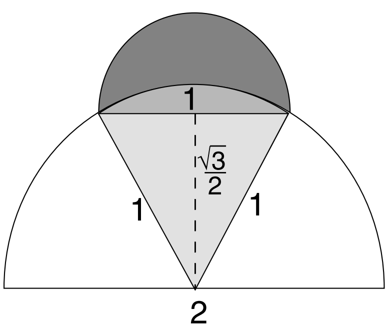
$$\frac{1}{2}\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\pi\left(\frac{1}{2}\right)^2=\frac{\sqrt{3}}{4}+\frac{1}{8}\pi.$$

The area of the lune results from subtracting from this the area of the sector of the larger semicircle,

$$\frac{1}{6}\pi(1)^2=\frac{1}{6}\pi.$$

So the area of the lune is

$$\frac{\sqrt{3}}{4}+\frac{1}{8}\pi-\frac{1}{6}\pi=\frac{\sqrt{3}}{4}-\frac{1}{24}\pi.$$



Note that the answer does not depend on the position of the lune on the semicircle.

20. (E) The largest base-9 three-digit number is $9^3-1=728$ and the smallest base-11 three-digit number is $11^2=121$. There are 608 integers that satisfy $121\le n\le 728$, and 900 three-digit numbers altogether, so the probability is $608/900\approx 0.7$.

21. (D) The numbers of the three types of cookies must have a sum of six. Possible sets of whole numbers whose sum is six are

$$0, 0, 6;\ \ 0, 1, 5;\ \ 0, 2, 4;\ \ 0, 3, 3;\ \ 1, 1, 4;\ \ 1, 2, 3;\ \ \text{ and } 2, 2, 2.$$

Every ordering of each of these sets determines a different assortment of cookies. There are 3 orders for each of the sets

$$0, 0, 6;\ \ 0, 3, 3;\ \text{ and } 1, 1, 4.$$

There are 6 orders for each of the sets

$$0, 1, 5;\ \ 0, 2, 4;\ \text{ and } 1, 2, 3.$$

There is only one order for $2, 2, 2$. Therefore the total number of assortments of six cookies is $3 \cdot 3 + 3 \cdot 6 + 1 = 28$.

OR

Construct eight slots, six to place the cookies in and two to divide the cookies by type. Let the number of chocolate chip cookies be the number of slots to the left of the first divider, the number of oatmeal cookies be the number of slots between the two dividers, and the number of peanut butter cookies be the number of slots to the right of the second divider. For example, $111 \mid 11 \mid 1$ represents three chocolate chip cookies, two oatmeal cookies, and one peanut butter cookie. There are $\binom{8}{2} = 28$ ways to place the two dividers, so there are 28 ways to select the six cookies.

22. (B) We have $EA = 5$ and $CH = 3$. Triangles GCH and GEA are similar, so

$$\frac{GC}{GE} = \frac{3}{5} \quad \text{and} \quad \frac{CE}{GE} = \frac{GE - GC}{GE} = 1 - \frac{3}{5} = \frac{2}{5}.$$

Triangles GFE and CDE are similar, so

$$\frac{GF}{8} = \frac{CE}{GE} = \frac{5}{2}$$

and $FG = 20$.

OR

Place the figure in the coordinate plane with the origin at D, $\overline{DA}$ on the positive x-axis, and $\overline{DC}$ on the positive y-axis. Then $H = (3, 8)$ and $A = (9, 0)$, so line AG has the equation

$$y = -\frac{4}{3}x + 12.$$

Also, $C = (0, 8)$ and $E = (4, 0)$, so line EG has the equation

$$y = -2x + 8.$$

The lines intersect at $(-6, 20)$, so $FG = 20$.

23. (C) The base row of the large equilateral triangle has 1001 triangles pointing downward and 1002 pointing upward. This base row requires 3(1002) toothpicks since the downward pointing triangles require no additional toothpicks. Each succeeding row will require one less set of 3 toothpicks, so the total number of toothpicks required is

$$3(1002 + 1001 + 1000 + \cdots + 2 + 1) = 3 \cdot \frac{1002 \cdot 1003}{2} = 1,507,509.$$

OR

Create a table:

Number of Rows	Number of Triangles in Base Row	Number of Toothpicks in All Rows
1	1	3
2	3	$3+6$
3	5	$3+6+9$
$\vdots$	$\vdots$	$\vdots$
n	$2n-1$	$3(1+2+\cdots+n)$

Thus

$$2003 = 2n - 1 \quad \text{so} \quad n = 1002.$$

The number of toothpicks is

$$3(1 + 2 + \cdots + 1002) = 3\frac{(1002)(1003)}{2} = 1,507,509.$$

24. (E) Let R1, ..., R5 and B3, ..., B6 denote the numbers on the red and blue cards, respectively. Note that R4 and R5 divide evenly into only B4 and B5, respectively. Thus the stack must be R4, B4, ..., B5, R5, or the reverse. Since R2 divides evenly into only B4 and B6, we must have R4, B4, R2, B6, ..., B5, R5, or the reverse. Since R3 divides evenly into only B3 and B6, the stack must be R4, B4, R2, B6, R3, B3, R1, B5, R5, or the reverse. In either case, the sum of the middle three cards is 12.

25. (B) Note that $n = 100q + r = q + r + 99q$. Hence $q + r$ is divisible by 11 if and only if n is divisible by 11. Since $10,000 \le n \le 99,999$, there are

$$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$$

such numbers.

The

American Mathematics Contest 12 (AMC 12)

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