The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

5th Annual American Mathematics Contest 10

AMC 10 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 25, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

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1. **(C)** There are $22 - 12 + 1 = 11$ reserved rows. Because there are 33 seats in each row, there are $(33)(11) = 363$ reserved seats.

2. **(B)** There are 10 two-digit numbers with a 7 as their 10's digit, and 9 two-digit numbers with 7 as their units digit. Because 77 satisfies both of these properties, the answer is $10 + 9 - 1 = 18$.

3. **(A)** At Jenny's fourth practice she made $\frac{1}{2}(48) = 24$ free throws. At her third practice she made 12, at her second practice she made 6, and at her first practice she made 3.

4. **(B)** Since $6! = 720 = 2^4 \cdot 3^2 \cdot 5$, the prime factors of P can consist of at most 2's, 3's, and 5's. The least possible number of 2's is two, which occurs when 4 is not visible. The least possible number of 3's is one, which occurs when either 3 or 6 is not visible, and the least number of 5's is zero, when 5 is not visible. Thus P must be divisible by $2^2 \cdot 3 = 12$, but not necessarily by any larger number.

5. **(D)** If $d \neq 0$, the value of the expression can be increased by interchanging 0 with the value of d. Therefore the maximum value must occur when $d = 0$. If $a = 1$, the value is c, which is 2 or 3. If $b = 1$, the value is $c \cdot a = 6$. If $c = 1$, the value is a^b, which is $2^3 = 8$ or $3^2 = 9$. Thus the maximum value is 9.

6. **(C)** Note that for $m < n$ we have

$$m! \cdot n! = (m!)^2 \cdot (m+1) \cdot (m+2) \cdot \cdots \cdot n.$$

Therefore $m! \cdot n!$ is a perfect square if and only if

$$(m+1) \cdot (m+2) \cdot \cdots \cdot n$$

is a perfect square. For the five answer choices, that quantity is

$$99,\ 99 \cdot 100,\ 100,\ 100 \cdot 101,\ \text{and}\ \ 101,$$

and of those only 100 is a perfect square. Therefore the answer is $99! \cdot 100!$.

7. **(A)** Isabella received $10d/7$ Canadian dollars at the border and spent 60 of them. Thus $10d/7 - 60 = d$, from which it follows that $d = 140$, and the sum of the digits of d is 5.

8. **(A)** Let downtown St. Paul, downtown Minneapolis, and the airport be located at S, M, and A, respectively. Then $\triangle MAS$ has a right angle at A, so by the Pythagorean Theorem,

$$MS = \sqrt{10^2 + 8^2} = \sqrt{164} \approx \sqrt{169} = 13.$$

9. **(B)** The areas of the regions enclosed by the square and the circle are $10^2 = 100$ and $\pi(10)^2 = 100\pi$, respectively. One quarter of the second region is also included in the first, so the area of the union is

$$100 + 100\pi - 25\pi = 100 + 75\pi.$$

10. **(D)** If there are n rows in the display, the bottom row contains $2n-1$ cans. The total number of cans is therefore the sum of the arithmetic series

$$1+3+5+\cdots+(2n-1),$$

which is

$$\frac{n}{2}[(2n-1)+1]=n^2.$$

Thus $n^2=100$, so $n=10$.

11. **(C)** There are $8\cdot 8=64$ ordered pairs that can represent the top numbers on the two dice. Let m and n represent the top numbers on the dice. Then $mn>m+n$ implies that $mn-m-n>0$, that is,

$$1<mn-m-n+1=(m-1)(n-1).$$

This inequality is satisfied except when $m=1$, $n=1$, or when $m=n=2$. There are 16 ordered pairs (m,n) excluded by these conditions, so the probability that the product is greater than the sum is

$$\frac{64-16}{64}=\frac{48}{64}=\frac{3}{4}.$$

12. **(A)** The area of the annulus is the difference between the areas of the two circles, which is $\pi b^2-\pi c^2$. Because the tangent $\overline{XZ}$ is perpendicular to the radius $\overline{OZ}$, $b^2-c^2=a^2$, so the area is πa^2.

13. **(B)** The height in millimeters of any stack with an odd number of coins has a 5 in the hundredth place. The height of any two coins has an odd digit in the tenth place and a zero in the hundredth place. Therefore any stack with zeros in both its tenth and hundredth places must consist of a number of coins that is a multiple of 4. The highest stack of 4 coins has a height of $4(1.95)=7.8$ mm, and the shortest stack of 12 coins has a height of $12(1.35)=16.2$ mm, so no number other than 8 can work. Note that a stack of 8 quarters has a height of $8(1.75)=14$ mm.

14. **(C)** If there are initially B blue marbles in the bag, after red marbles are added, then the total number of marbles in the bag must be $3B$. Then after the yellow marbles are added, the number of marbles in the bag must be $5B$. Finally, adding B blue marbles to the bag gives $2B$ blue marbles out of $6B$ total marbles. Thus $1/3$ of the marbles are blue.

OR

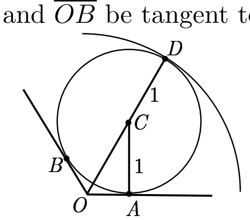
Just before the number of blue marbles is doubled, the ratio of blue marbles to non-blue marbles is 1 to 4. Doubling the number of blue marbles makes the ratio 2 to 4, so $1/3$ of the marbles are blue.

15. **(A)** Because the value of Patty's money would increase if the dimes and nickels were interchanged, she must have more nickels than dimes. Interchanging one nickel for a dime increases the amount by 5 cents, so she has $70/5 = 14$ more nickels than dimes. Therefore she has

$$\frac{1}{2}(20-14) = 3 \quad \text{dimes} \quad \text{and} \quad 20-3=17 \quad \text{nickels},$$

and her coins are worth $3 \cdot 10 + 17 \cdot 5 = 115$ cents $= \$1.15$.

16. **(D)** Let O be the center of the large circle, let C be the center of one of the small circles, and let $\overline{OA}$ and $\overline{OB}$ be tangent to the small circle at A and B.



By symmetry, $\angle AOB = 120°$ and $\angle AOC = 60°$. Thus $\triangle AOC$ is a 30-60-90 degree right triangle, and $AC = 1$, so

$$OC = \frac{2}{\sqrt{3}} AC = \frac{2\sqrt{3}}{3}.$$

If OD is a radius of the large circle through C, then

$$OD = CD + OC = 1 + \frac{2\sqrt{3}}{3} = \frac{3+2\sqrt{3}}{3}.$$

17. **(B)** Let Jack's age be $10x+y$ and Bill's age be $10y+x$. In five years Jack will be twice as old as Bill. Therefore

$$10x + y + 5 = 2(10y + x + 5),$$

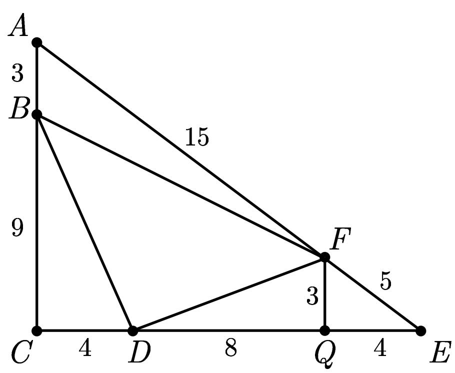
so $8x = 19y + 5$. The expression $19y + 5 = 16y + 8 + 3(y-1)$ is a multiple of 8 if and only if $y-1$ is a multiple of 8. Since both x and y are 9 or less, the only solution is $y = 1$ and $x = 3$. Thus Jack is 31 and Bill is 13, so the difference between their ages is 18.

18. **(E)** The area of $\triangle ACE$ is $(1/2)(12)(16) = 96$. Draw $\overline{FQ} \perp \overline{CE}$. By similar triangles, $FQ = 3$ and $QE = 4$. The area of trapezoid $BCQF$ is $(1/2)(3 + 9)(12) = 72$. Since $\triangle BCD$ and $\triangle FDQ$ have areas 18 and 12, respectively, the area of $\triangle BDF$ is $72 - 18 - 12 = 42$. The desired ratio is $42/96 = 7/16$.

OR

Note that each of $\triangle ABF$, $\triangle BCD$, and $\triangle DEF$ has a base-altitude pair where the base and altitude are, respectively, 3/4 and 1/4 that of a corresponding base and altitude for $\triangle ACE$. Hence

$$\frac{\text{Area of} \triangle BDF}{\text{Area of} \triangle ACE} = 1 - 3(1/4)(3/4) = 7/16.$$



19. **(C)** Let a_k be the k^{th} term of the sequence. For $k \geq 3$,

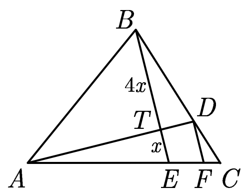
$$a_{k+1} = a_{k-2} + a_{k-1} - a_k, \quad \text{so} \quad a_{k+1} - a_{k-1} = -(a_k - a_{k-2}).$$

Because the sequence begins

$$2001, 2002, 2003, 2000, 2005, 1998, \ldots,$$

it follows that the odd-numbered terms and the even-numbered terms each form arithmetic progressions with common differences of 2 and -2, respectively. The 2004^{th} term of the original sequence is the 1002^{nd} term of the sequence 2002, 2000, 1998,…, and that term is $2002 + 1001(-2) = 0$.

20. **(D)** Let F be a point on $\overline{AC}$ such that $\overline{DF}$ is parallel to $\overline{BE}$. Let $BT = 4x$ and $ET = x$.



Because $\triangle ATE$ and $\triangle ADF$ are similar, we have

$$\frac{DF}{x} = \frac{AD}{AT} = \frac{4}{3}, \quad \text{and} \quad DF = \frac{4x}{3}.$$

Also, $\triangle BEC$ and $\triangle DFC$ are similar, so

$$\frac{CD}{BC} = \frac{DF}{BE} = \frac{4x/3}{5x} = \frac{4}{15}.$$

Thus

$$\frac{CD}{BD} = \frac{CD/BC}{1-(CD/BC)} = \frac{4/15}{1-4/15} = \frac{4}{11}.$$

OR

Let $s = \text{Area}(\triangle ABC)$. Then

$$\text{Area}(\triangle TBC) = \frac{1}{4}s \quad \text{and} \quad \text{Area}(\triangle ATC) = \frac{1}{5}s,$$

so

$$\text{Area}(\triangle ATB) = \text{Area}(\triangle ABC) - \text{Area}(\triangle TBC) - \text{Area}(\triangle ATC) = \frac{11}{20}s.$$

Hence

$$\frac{CD}{BD} = \frac{\text{Area}(\triangle ADC)}{\text{Area}(\triangle ABD)} = \frac{\text{Area}(\triangle ATC)}{\text{Area}(\triangle ATB)} = \frac{s/5}{11s/20} = \frac{4}{11}.$$

21. **(A)** The smallest number that appears in both sequences is 16. The two sequences have common differences 3 and 7, whose least common multiple is 21, so a number appears in both sequences if and only if it is in the form

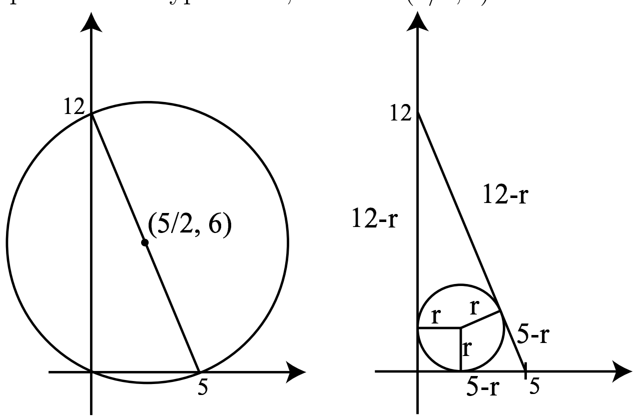
$$16 + 21k,$$

where k is a nonnegative integer. Such a number is in the first 2004 terms of both sequences if and only if

$$16 + 21k \le 1 + 2003(3) = 6010.$$

Thus $0 \le k \le 285$, so there are 286 duplicate numbers. Therefore the number of distinct numbers is $4008 - 286 = 3722$.

22. **(D)** The triangle is a right triangle that can be placed in a coordinate system with vertices at $(0,0)$, $(5,0)$, and $(0,12)$. The center of the circumscribed circle is the midpoint of the hypotenuse, which is $(5/2,6)$.



To determine the radius r of the inscribed circle notice that the hypotenuse of the triangle is

$$(12-r)+(5-r)=13 \quad \text{so} \quad r=2.$$

So the center of the inscribed circle is $(2,2)$, and the distance between the two centers is

$$\sqrt{\left(\frac{5}{2}-2\right)^2+(6-2)^2}=\frac{\sqrt{65}}{2}.$$

23. **(B)** If the orientation of the cube is fixed, there are $2^6=64$ possible arrangements of colors on the faces. There are

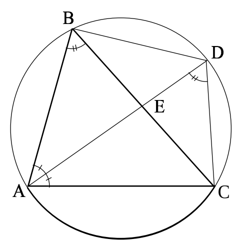
$$2\binom{6}{6}=2$$

arrangements in which all six faces are the same color and

$$2\binom{6}{5}=12$$

arrangements in which exactly five faces have the same color. In each of these cases the cube can be placed so that the four vertical faces have the same color. The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are $2(3)=6$ such arrangements. The total number of suitable arrangements is therefore $2+12+6=20$, and the probability is $20/64=5/16$.

24. **(B)** Suppose that AD and BC intersect at E.



Since $\angle ADC$ and $\angle ABC$ cut the same arc of the circumscribed circle, the Inscribed Angle Theorem implies that

$$\angle ABC = \angle ADC.$$

Also, $\angle EAB = \angle CAD$, so $\triangle ABE$ is similar to $\triangle ADC$, and

$$\frac{AD}{CD} = \frac{AB}{BE}.$$

By the Angle Bisector Theorem,

$$\frac{BE}{EC} = \frac{AB}{AC},$$

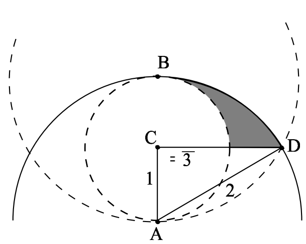
so

$$BE = \frac{AB}{AC} \cdot EC = \frac{AB}{AC}(BC - BE) \quad \text{and} \quad BE = \frac{AB \cdot BC}{AB + AC}.$$

Hence

$$\frac{AD}{CD} = \frac{AB}{BE} = \frac{AB + AC}{BC} = \frac{7 + 8}{9} = \frac{5}{3}.$$

25. **(B)** The centers of the two larger circles are at A and B. Let C be the center of the smaller circle, and let D be one of the points of intersection of the two larger circles.



Then $\triangle ACD$ is a right triangle with $AC = 1$ and $AD = 2$, so $CD = \sqrt{3}$, $\angle CAD = 60°$, and the area of $\triangle ACD$ is $\sqrt{3}/2$. The area of 1/4 of the shaded region, as shown in the Figure, is the area of sector BAD of the circle centered at A, minus the area of $\triangle ACD$, minus the area of 1/4 of the smaller circle. That area is

$$\frac{2}{3}\pi - \frac{\sqrt{3}}{2} - \frac{1}{4}\pi = \frac{5}{12}\pi - \frac{\sqrt{3}}{2},$$

so the area of the entire shaded region is

$$4\left(\frac{5}{12}\pi - \frac{\sqrt{3}}{2}\right) = \frac{5}{3}\pi - 2\sqrt{3}.$$

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