Tuesday, FEBRUARY 1, 2005

6th Annual American Mathematics Contest 10

AMC 10

Contest A

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 2.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 120 or above or finish in the top 1% on this AMC 10 will be invited to take the 23rd annual American Invitational Mathematics Examination (AIME) on Tuesday, March 8, 2005 or Tuesday, March 22, 2005. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.

1. While eating out, Mike and Joe each tipped their server \$2. Mike tipped 10% of his bill and Joe tipped 20% of his bill. What was the difference, in dollars, between their bills?

   (A) 2 **(B)** 4 **(C)** 5 **(D)** 10 **(E)** 20

2. For each pair of real numbers $a \neq b$, define the operation $\star$ as

$$(a \star b) = \frac{a+b}{a-b}.$$

   What is the value of $((1 \star 2) \star 3)$?

   (A) $-\frac{2}{3}$ **(B)** $-\frac{1}{5}$ **(C)** 0 **(D)** $\frac{1}{2}$ **(E)** This value is not defined.

3. The equations $2x + 7 = 3$ and $bx - 10 = -2$ have the same solution x. What is the value of b?

   (A) -8 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 8

4. A rectangle with a diagonal of length x is twice as long as it is wide. What is the area of the rectangle?

   (A) $\frac{1}{4}x^2$ **(B)** $\frac{2}{5}x^2$ **(C)** $\frac{1}{2}x^2$ **(D)** x^2 **(E)** $\frac{3}{2}x^2$

5. A store normally sells windows at \$100 each. This week the store is offering one free window for each purchase of four. Dave needs seven windows and Doug needs eight windows. How many dollars will they save if they purchase the windows together rather than separately?

   (A) 100 **(B)** 200 **(C)** 300 **(D)** 400 **(E)** 500

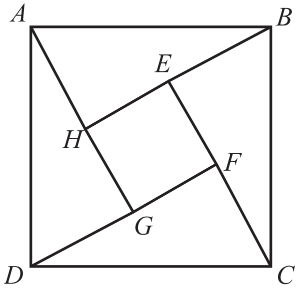
6. The average (mean) of 20 numbers is 30, and the average of 30 other numbers is 20. What is the average of all 50 numbers?

   (A) 23 **(B)** 24 **(C)** 25 **(D)** 26 **(E)** 27

7. Josh and Mike live 13 miles apart. Yesterday Josh started to ride his bicycle toward Mike's house. A little later Mike started to ride his bicycle toward Josh's house. When they met, Josh had ridden for twice the length of time as Mike and at four-fifths of Mike's rate. How many miles had Mike ridden when they met?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

8. In the figure, the length of side AB of square $ABCD$ is $\sqrt{50}$, E is between B and H, and $BE = 1$. What is the the area of the inner square $EFGH$?



(A) 25 **(B)** 32 **(C)** 36 **(D)** 40 **(E)** 42

9. Three tiles are marked X and two other tiles are marked O. The five tiles are randomly arranged in a row. What is the probability that the arrangement reads XOXOX?

(A) $\frac{1}{12}$ **(B)** $\frac{1}{10}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{3}$

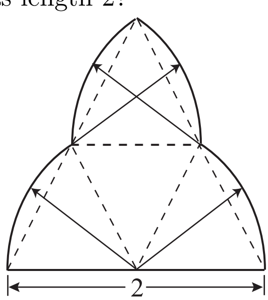
10. There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

(A) -16 **(B)** -8 **(C)** 0 **(D)** 8 **(E)** 20

11. A wooden cube n units on a side is painted red on all six faces and then cut into n^3 unit cubes. Exactly one-fourth of the total number of faces of the unit cubes are red. What is n?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

12. The figure shown is called a *trefoil* and is constructed by drawing circular sectors about sides of the congruent equilateral triangles. What is the area of a trefoil whose horizontal base has length 2?



(A) $\frac{1}{3}\pi + \frac{\sqrt{3}}{2}$ **(B)** $\frac{2}{3}\pi$ **(C)** $\frac{2}{3}\pi + \frac{\sqrt{3}}{4}$ **(D)** $\frac{2}{3}\pi + \frac{\sqrt{3}}{3}$ **(E)** $\frac{2}{3}\pi + \frac{\sqrt{3}}{2}$

13. How many positive integers n satisfy the following condition:

$$(130n)^{50} > n^{100} > 2^{200} \ ?$$

(A) 0 **(B)** 7 **(C)** 12 **(D)** 65 **(E)** 125

14. How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

(A) 41 **(B)** 42 **(C)** 43 **(D)** 44 **(E)** 45

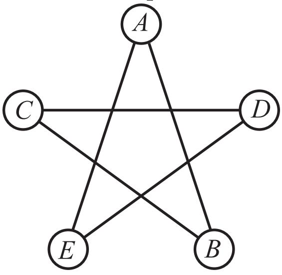
15. How many positive cubes divide $3! \cdot 5! \cdot 7!$?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

16. The sum of the digits of a two-digit number is subtracted from the number. The units digit of the result is 6. How many two-digit numbers have this property?

(A) 5 **(B)** 7 **(C)** 9 **(D)** 10 **(E)** 19

17. In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?

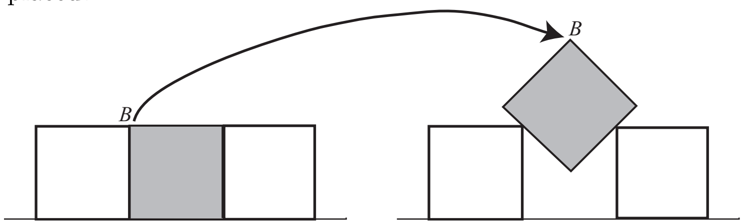


(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

18. Team A and team B play a series. The first team to win three games wins the series. Each team is equally likely to win each game, there are no ties, and the outcomes of the individual games are independent. If team B wins the second game and team A wins the series, what is the probability that team B wins the first game?

(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

19. Three one-inch squares are placed with their bases on a line. The center square is lifted out and rotated 45°, as shown. Then it is centered and lowered into its original location until it touches both of the adjoining squares. How many inches is the point B from the line on which the bases of the original squares were placed?



(A) 1 **(B)** $\sqrt{2}$ **(C)** $\frac{3}{2}$ **(D)** $\sqrt{2}+\frac{1}{2}$ **(E)** 2

20. An equiangular octagon has four sides of length 1 and four sides of length $\sqrt{2}/2$, arranged so that no two consecutive sides have the same length. What is the area of the octagon?

(A) $\frac{7}{2}$ **(B)** $\frac{7\sqrt{2}}{2}$ **(C)** $\frac{5+4\sqrt{2}}{2}$ **(D)** $\frac{4+5\sqrt{2}}{2}$ **(E)** 7

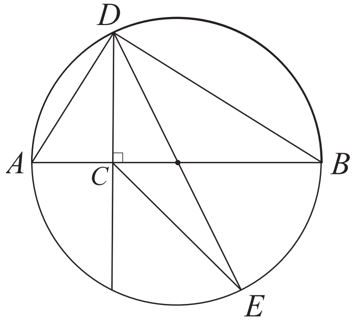
21. For how many positive integers n does $1+2+\cdots+n$ evenly divide $6n$?

(A) 3 **(B)** 5 **(C)** 7 **(D)** 9 **(E)** 11

22. Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

(A) 166 **(B)** 333 **(C)** 500 **(D)** 668 **(E)** 1001

23. Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$

24. For each positive integer $m > 1$, let $P(m)$ denote the greatest prime factor of m. For how many positive integers n is it true that both $P(n) = \sqrt{n}$ and $P(n+48) = \sqrt{n+48}$?

(A) 0 **(B)** 1 **(C)** 3 **(D)** 4 **(E)** 5

25. In $\triangle ABC$ we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on $\overline{AB}$ and $\overline{AC}$ respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

(A) $\frac{266}{1521}$ **(B)** $\frac{19}{75}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{19}{56}$ **(E)** 1

WRITE TO US!

Correspondence about the problems and solutions for this AMC 10 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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2005 AIME

The AIME will be held on Tuesday, March 8, 2005 with the alternate on March 22, 2005. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 19 and 20, 2005. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

MINIMUM ORDER: $10 (before shipping/handling fee), *PAYMENT IN U.S. FUNDS ONLY made payable to the* American Mathematics Competitions or VISA/MASTERCARD accepted. Include card number, expiration date, cardholder name, address, telephone and email. U.S.A. and Canadian orders must be prepaid and will be shipped Priority Mail, UPS or Air Mail.
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Examinations: Each price is for one copy of an exam and its solutions. Specify the years you want and how many copies of each. All prices effective to September 1, 2005.

- AMC 10 2000–2005/(AHSME) AMC 12 1989–2005, $1 per exam copy.
- AIME 1983–1993, 1995–2005, $2 per copy per year (2005 available after March).
- USA and International Math Olympiads, 1989–1999, $5 per copy per year, (quantities limited)
- National Summary of Results and Awards, 1989–2005, $10 per copy per year.
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2005

AMC 10 – Contest A

DO NOT OPEN UNTIL TUESDAY, FEBRUARY 1, 2005

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHER'S MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE FEBRUARY 1.** Nothing is needed from inside this package until February 1.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

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