The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

6th Annual American Mathematics Contest 10

AMC 10 – Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 1, 2005

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions for this AMC 10 and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **(D)** Since Mike tipped \$2, which was $10\% = 1/10$ of his bill, his bill must have been $2 \cdot 10 = 20$ dollars. Similarly, Joe tipped \$2, which was $20\% = 1/5$ of his bill, so his bill must have been $2 \cdot 5 = 10$ dollars. The difference between their bills is therefore \$10.

2. **(C)** First we have
$$(1 \star 2) = \frac{1+2}{1-2} = -3.$$
Then
$$((1 \star 2) \star 3) = (-3 \star 3) = \frac{-3+3}{-3-3} = \frac{0}{-6} = 0.$$

3. **(B)** Since $2x + 7 = 3$ we have $x = -2$. Hence
$$-2 = bx - 10 = -2b - 10, \quad \text{so} \quad 2b = -8, \text{ and } b = -4.$$

4. **(B)** Let w be the width of the rectangle. Then the length is $2w$, and
$$x^2 = w^2 + (2w)^2 = 5w^2.$$
The area is consequently $w(2w) = 2w^2 = \frac{2}{5}x^2$.

5. **(A)** If Dave buys seven windows separately he will purchase six and receive one free, for a cost of \$600. If Doug buys eight windows separately, he will purchase seven and receive one free, for a total cost of \$700. The total cost to Dave and Doug purchasing separately will be \$1300. If they purchase fifteen windows together, they will need to purchase only 12 windows, for a cost of \$1200, and will receive 3 free. This will result in a savings of \$100.

6. **(B)** The sum of the 50 numbers is $20 \cdot 30 + 30 \cdot 20 = 1200$. Their average is $1200/50 = 24$.

7. **(B)** Because (rate)(time) = (distance), the distance Josh rode was $(4/5)(2) = 8/5$ of the distance that Mike rode. Let m be the number of miles that Mike had ridden when they met. Then the number of miles between their houses is
$$13 = m + \frac{8}{5}m = \frac{13}{5}m.$$
Thus $m = 5$.

8. **(C)** The symmetry of the figure implies that $\triangle ABH$, $\triangle BCE$, $\triangle CDF$, and $\triangle DAG$ are congruent right triangles. So
$$BH = CE = \sqrt{BC^2 - BE^2} = \sqrt{50 - 1} = 7,$$
and $EH = BH - BE = 7 - 1 = 6$. Hence the square $EFGH$ has area $6^2 = 36$.

OR

As in the first solution, $BH = 7$. Now note that $\triangle ABH$, $\triangle BCE$, $\triangle CDF$, and $\triangle DAG$ are congruent right triangles, so
$$\text{Area}(EFGH) = \text{Area}(ABCD) - 4\text{Area}(\triangle ABH) = 50 - 4\left(\tfrac{1}{2} \cdot 1 \cdot 7\right) = 36.$$

9. **(B)** There are three X's and two O's, and the tiles are selected without replacement, so the probability is

$$\frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{1} = \frac{1}{10}.$$

OR

The three tiles marked X are equally likely to lie in any of $\binom{5}{3} = 10$ positions, so the probability of this arrangement is $1/10$.

10. **(A)** The quadratic formula yields

$$x = \frac{-(a+8) \pm \sqrt{(a+8)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4}.$$

The equation has only one solution precisely when the value of the discriminant, $(a+8)^2 - 144$, is 0. This implies that $a = -20$ or $a = 4$, and the sum is -16.

OR

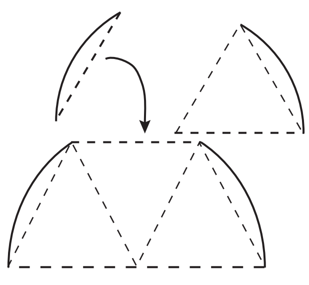
The equation has one solution if and only if the polynomial is the square of a binomial with linear term $\pm\sqrt{4x^2} = \pm 2x$ and constant term $\pm\sqrt{9} = \pm 3$. Because $(2x \pm 3)^2$ has a linear term $\pm 12x$, it follows that $a + 8 = \pm 12$. Thus a is either -20 or 4, and the sum of those values is -16.

11. **(B)** The unit cubes have a total of $6n^3$ faces, of which $6n^2$ are red. Therefore

$$\frac{1}{4} = \frac{6n^2}{6n^3} = \frac{1}{n}, \quad \text{so} \quad n = 4.$$

12. **(B)** The trefoil is constructed of four equilateral triangles and four circular segments, as shown. These can be combined to form four 60° circular sectors. Since the radius of the circle is 1, the area of the trefoil is

$$\frac{4}{6}\left(\pi \cdot 1^2\right) = \frac{2}{3}\pi.$$



13. **(E)** The condition is equivalent to

$$130n > n^2 > 2^4 = 16, \quad \text{so} \quad 130n > n^2 \quad \text{and} \quad n^2 > 16.$$

This implies that $130 > n > 4$. So n can be any of the 125 integers strictly between 130 and 4.

14. **(E)** The first and last digits must be both odd or both even for their average to be an integer. There are $5 \cdot 5 = 25$ odd-odd combinations for the first and last digits. There are $4 \cdot 5 = 20$ even-even combinations that do not use zero as the first digit. Hence the total is 45.

15. **(E)** Written as a product of primes, we have

$$3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

A cube that is a factor has a prime factorization of the form $2^p \cdot 3^q \cdot 5^r \cdot 7^s$, where p, q, r, and s are all multiples of 3. There are 3 possible values for p, which are 0, 3, and 6. There are 2 possible values for q, which are 0 and 3. The only value for r and for s is 0. Hence there are $6 = 3 \cdot 2 \cdot 1 \cdot 1$ distinct cubes that divide $3! \cdot 5! \cdot 7!$. They are

$$\begin{aligned} 1 =& 2^0 3^0 5^0 7^0, \quad 8 = 2^3 3^0 5^0 7^0, \quad 27 = 2^0 3^3 5^0 7^0, \\ 64 =& 2^6 3^0 5^0 7^0, \quad 216 = 2^3 3^3 5^0 7^0, \quad \text{and} \quad 1728 = 2^6 3^3 5^0 7^0. \end{aligned}$$

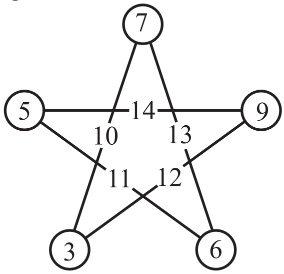
16. **(D)** Let $10a + b$ be the two-digit number. When $a + b$ is subtracted the result is $9a$. The only two-digit multiple of 9 that ends in 6 is $9 \cdot 4 = 36$, so $a = 4$. The ten numbers between 40 and 49, inclusive, have this property.

17. **(D)** Each number appears in two sums, so the sum of the sequence is

$$2(3 + 5 + 6 + 7 + 9) = 60.$$

The middle term of a five-term arithmetic sequence is the mean of its terms, so $60/5 = 12$ is the middle term.

The figure shows an arrangement of the five numbers that meets the requirement.



18. **(A)**

There are four possible outcomes,

$$\text{ABAA}, \quad \text{ABABA}, \quad \text{ABBAA}, \quad \text{and} \quad \text{BBAAA},$$

but they are not equally likely. This is because, in general, the probability of any specific four-game series is $(1/2)^4 = 1/16$, whereas the probability of any specific five-game series is $(1/2)^4 = 1/32$. Thus the first listed outcome is twice as likely as each of the other three. Let p be the probability of the occurrence $ABBAA$. Then the probability of $ABABA$ is also p, as is the probability of $BBAAA$, whereas the probability of $ABAA$ is $2p$. So

$$2p + p + p + p = 1, \quad \text{and} \quad p = \frac{1}{5}.$$

The only outcome in which team B wins the first game is $BBAAA$, so the probability of this occurring is $1/5$.

OR

To consider equally-likely cases, suppose that all five games are played, even if team A has won the series before the fifth game. Then the possible ways that team A can win the series, given that team B wins the second game, are

$$\text{BBAAA}, \quad \text{ABBAA}, \quad \text{ABABA}, \quad \text{ABAAB}, \quad \text{and} \quad \text{ABAAA}.$$

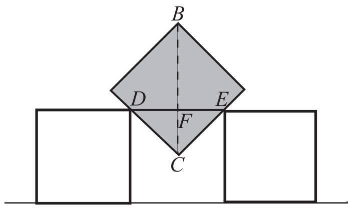
In only the first case does team B win the first game, so the probability of this occurring is $1/5$.

19. **(D)** Consider the rotated middle square shown in the figure. It will drop until length DE is 1 inch. Thus

$$FC = DF = FE = \frac{1}{2} \quad \text{and} \quad BC = \sqrt{2}.$$

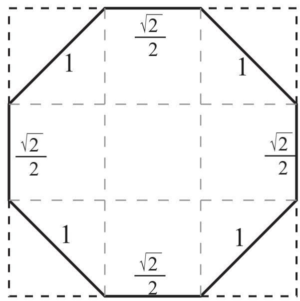
Hence $BF = \sqrt{2} - 1/2$. This is added to the 1 inch height of the supporting squares, so the overall height of point B above the line is

$$1 + BF = \sqrt{2} + \frac{1}{2} \text{ inches.}$$



20. **(A)** The octagon can be partitioned into five squares and four half squares, each with side length $\sqrt{2}/2$, so its area is

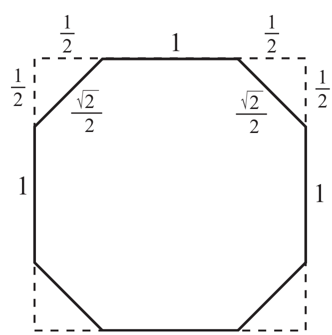
$$\left(5 + 4 \cdot \frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{7}{2}.$$



OR

The octagon can be obtained by removing four isosceles right triangles with legs of length 1/2 from a square with sides of length 2. Thus its area is

$$2^2 - 4 \cdot \frac{1}{2}\left(\frac{1}{2}\right)^2 = \frac{7}{2}.$$



21. **(B)** Because

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

$1 + 2 + \cdots + n$ divides the positive integer $6n$ if and only if

$$\frac{6n}{n(n+1)/2} = \frac{12}{n+1} \text{ is an integer.}$$

There are five such positive values of n, namely, 1, 2, 3, 5, and 11.

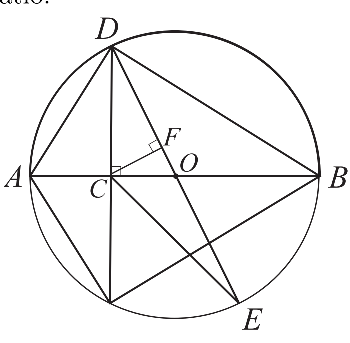
22. **(D)** The sets S and T consist, respectively, of the positive multiples of 4 that do not exceed $2005 \cdot 4 = 8020$ and the positive multiples of 6 that do not exceed $2005 \cdot 6 = 12{,}030$. Thus $S \bigcap T$, the set of numbers that are common to S and to T, consists of the positive multiples of 12 that do not exceed 8020. Let $\lfloor x \rfloor$ represent the largest integer that is less than or equal to x. Then the number of elements in the set $S \bigcap T$ is

$$\left\lfloor \frac{8020}{12} \right\rfloor = \left\lfloor 668 + \frac{1}{3} \right\rfloor = 668.$$

23. **(C)** Let O be the center of the circle. Each of $\triangle DCE$ and $\triangle ABD$ has a diameter of the circle as a side. Thus the ratio of their areas is the ratio of the two altitudes to the diameters. These altitudes are $\overline{DC}$ and the altitude from C to $\overline{DO}$ in $\triangle DCE$. Let F be the foot of this second altitude. Since $\triangle CFO$ is similar to $\triangle DCO$,

$$\frac{CF}{DC} = \frac{CO}{DO} = \frac{AO - AC}{DO} = \frac{\frac{1}{2}AB - \frac{1}{3}AB}{\frac{1}{2}AB} = \frac{1}{3},$$

which is the desired ratio.



OR

Because $AC = AB/3$ and $AO = AB/2$, we have $CO = AB/6$. Triangles DCO and DAB have a common altitude to $\overline{AB}$ so the area of $\triangle DCO$ is $\frac{1}{6}$ the area of $\triangle ADB$. Triangles DCO and ECO have equal areas since they have a common base $\overline{CO}$ and their altitudes are equal. Thus the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$ is $1/3$.

24. **(B)** The conditions imply that both n and $n+48$ are squares of primes. So for each successful value of n we have primes p and q with $p^2 = n+48$ and $q^2 = n$, and

$$48 = p^2 - q^2 = (p+q)(p-q).$$

The pairs of factors of 48 are

$$48 \text{ and } 1, \quad 24 \text{ and } 2, \quad 16 \text{ and } 3, \quad 12 \text{ and } 4, \quad \text{and} \quad 8 \text{ and } 6.$$

These give pairs (p,q), respectively, of

$$\left(\frac{49}{2}, \frac{47}{2}\right), \quad (13,11), \quad \left(\frac{19}{2}, \frac{13}{2}\right), \ (8,4), \quad \text{and} \quad (7,1).$$

Only $(p,q) = (13,11)$ gives prime values for p and for q, with $n = 11^2 = 121$ and $n+48 = 13^2 = 169$.

25. **(D)** We have

$$\frac{\text{Area}(ADE)}{\text{Area}(ABE)} = \frac{AD}{AB} = \frac{19}{25} \quad \text{and} \quad \frac{\text{Area}(ABE)}{\text{Area}(ABC)} = \frac{AE}{AC} = \frac{14}{42} = \frac{1}{3},$$

so

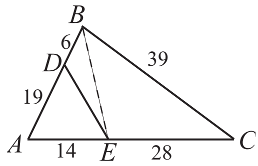
$$\frac{\text{Area}(ABC)}{\text{Area}(ADE)} = \frac{25}{19} \cdot \frac{3}{1} = \frac{75}{19},$$

and

$$\frac{\text{Area}(BCED)}{\text{Area}(ADE)} = \frac{\text{Area}(ABC) - \text{Area}(ADE)}{\text{Area}(ADE)} = \frac{75}{19} - 1 = \frac{56}{19}.$$

Thus

$$\frac{\text{Area}(ADE)}{\text{Area}(BCED)} = \frac{19}{56}.$$



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