The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

6th Annual American Mathematics Contest 10

AMC 10 – Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 16, 2005

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions for this AMC 10 and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **(A)** The scouts bought $1000/5 = 200$ groups of 5 candy bars at a total cost of $200 \cdot 2 = 400$ dollars. They sold $1000/2 = 500$ groups of 2 candy bars for a total of $500 \cdot 1 = 500$ dollars. Their profit was $\$500 - \$400 = \$100$.

2. **(D)** We have
$$\frac{x}{100} \cdot x = 4, \quad \text{so} \quad x^2 = 400.$$
Because $x > 0$, it follows that $x = 20$.

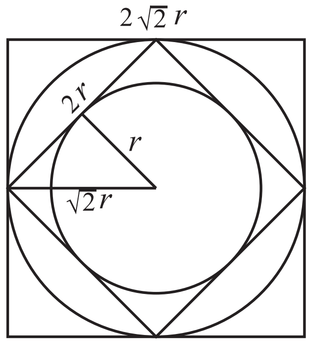
3. **(D)** After the first day,
$$1 - \frac{1}{3} = \frac{2}{3}$$
of the paint remains. On the second day,
$$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$
of the paint is used. So for the third day
$$1 - \frac{1}{3} - \frac{2}{9} = \frac{4}{9}$$
of the original gallon of paint is available.

4. **(D)** It follows from the definition that
$$\begin{aligned}(5 \diamond 12) \diamond ((-12) \diamond (-5)) &= \sqrt{5^2 + 12^2} \diamond \sqrt{(-12)^2 + (-5)^2} \\ &= 13 \diamond 13 = \sqrt{13^2 + 13^2} = 13\sqrt{2}.\end{aligned}$$

5. **(C)** The number of CDs that Brianna will finally buy is three times the number she has already bought. The fraction of her money that will be required for all the purchases is $(3)(1/5) = 3/5$. The fraction she will have left is $1 - 3/5 = 2/5$.

6. **(B)** To earn an A on at least 80% of her quizzes, Lisa needs to receive an A on at least $(0.8)(50) = 40$ quizzes. Thus she must earn an A on at least $40 - 22 = 18$ of the remaining 20. So she can earn a grade lower than an A on at most 2 of the remaining quizzes.

7. **(B)** Let the radius of the smaller circle be r. Then the side length of the smaller square is $2r$. The radius of the larger circle is half the length of the diagonal of the smaller square, so it is $\sqrt{2}r$. Hence the larger square has sides of length $2\sqrt{2}r$. The ratio of the area of the smaller circle to the area of the larger square is therefore
$$\frac{\pi r^2}{\left(2\sqrt{2}r\right)^2} = \frac{\pi}{8}.$$



8. **(A)** The four white quarter circles in each tile have the same area as a whole circle of radius $1/2$, that is, $\pi(1/2)^2 = \pi/4$ square feet. So the area of the shaded portion of each tile is $1 - \pi/4$ square feet. Since there are $8 \cdot 10 = 80$ tiles in the entire floor, the area of the total shaded region in square feet is

$$80\left(1 - \frac{\pi}{4}\right) = 80 - 20\pi.$$

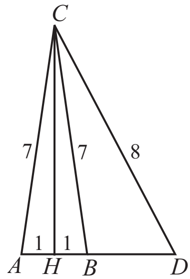
9. **(D)** An odd sum requires either that the first die is even and the second is odd or that the first die is odd and the second is even. The probability is

$$\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

10. **(A)** Let $\overline{CH}$ be an altitude of $\triangle ABC$. Applying the Pythagorean Theorem to $\triangle CHB$ and to $\triangle CHD$ produces

$$8^2 - (BD+1)^2 = CH^2 = 7^2 - 1^2 = 48, \quad \text{so} \quad (BD+1)^2 = 16.$$

Thus $BD = 3$.



11. **(E)** The sequence begins 2005, 133, 55, 250, 133, ... Thus after the initial term 2005, the sequence repeats the cycle 133, 55, 250. Because $2005 = 1 + 3 \cdot 668$, the 2005^{th} term is the same as the last term of the repeating cycle, 250.

12. **(E)** Exactly one die must have a prime face on top, and the other eleven must have 1's. The prime die can be any one of the twelve, and the prime can be 2, 3, or 5. Thus the probability of a prime face on any one die is 1/2, and the probability of a prime product is

$$12\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^{11} = \left(\frac{1}{6}\right)^{10}.$$

13. **(C)** Between 1 and 2005, there are 668 multiples of 3, 501 multiples of 4, and 167 multiples of 12. So there are

$$(668 - 167) + (501 - 167) = 835$$

numbers between 1 and 2005 that are integer multiples of 3 or of 4 but not of 12.

14. **(C)** Drop $\overline{MQ}$ perpendicular to $\overline{BC}$. Then $\triangle MQC$ is a 30–60–90° triangle, so $MQ = \sqrt{3}/2$, and the area of $\triangle CDM$ is

$$\frac{1}{2}\left(2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

OR

Triangles ABC and CDM have equal bases. Because M is the midpoint of $\overline{AC}$, the ratio of the altitudes from M and from A is 1/2. So the area of $\triangle CDM$ is half of the area of $\triangle ABC$. Since

$$\text{Area}(\triangle ABC) = \frac{\sqrt{3}}{4} \cdot 2^2 = \sqrt{3}, \quad \text{we have} \quad \text{Area}(\triangle CDM) = \frac{\sqrt{3}}{2}.$$

15. **(D)** There are

$$\binom{8}{2} = \frac{8!}{6! \cdot 2!} = 28$$

ways to choose the bills. A sum of at least \$20 is obtained by choosing both \$20 bills, one of the \$20 bills and one of the six smaller bills, or both \$10 bills. Hence the probability is

$$\frac{1 + 2 \cdot 6 + 1}{28} = \frac{14}{28} = \frac{1}{2}.$$

16. **(D)** Let r_1 and r_2 be the roots of $x^2+px+m=0$. Since the roots of $x^2+mx+n=0$ are $2r_1$ and $2r_2$, we have the following relationships:

$$m = r_1 r_2, \quad n = 4r_1 r_2, \quad p = -(r_1 + r_2), \quad \text{and} \quad m = -2(r_1 + r_2).$$

So

$$n = 4m, \quad p = \frac{1}{2}m, \quad \text{and} \quad \frac{n}{p} = \frac{4m}{\frac{1}{2}m} = 8.$$

OR

The roots of
$$\left(\frac{x}{2}\right)^2 + p\left(\frac{x}{2}\right) + m = 0$$
are twice those of $x^2 + px + m = 0$. Since the first equation is equivalent to $x^2 + 2px + 4m = 0$, we have
$$m = 2p \quad \text{and} \quad n = 4m, \quad \text{so} \quad \frac{n}{p} = 8.$$

17. **(B)** Because
$$4^{a\cdot b\cdot c\cdot d} = \left(\left(\left(4^a\right)^b\right)^c\right)^d = \left(\left(5^b\right)^c\right)^d = \left(6^c\right)^d = 7^d = 8 = 4^{3/2},$$
we have $a \cdot b \cdot c \cdot d = 3/2$.

18. **(D)** The last seven digits of the phone number use seven of the eight digits $\{2, 3, 4, 5, 6, 7, 8, 9\}$, so all but one of these digits is used. The unused digit can be chosen in eight ways. The remaining seven digits are then placed in increasing order to obtain a possible phone number. Thus there are 8 possible phone numbers.

19. **(B)** The percentage of students getting 95 points is
$$100 - 10 - 25 - 20 - 15 = 30,$$
so the mean score on the exam is
$$0.10(70) + 0.25(80) + 0.20(85) + 0.15(90) + 0.30(95) = 86.$$
Since fewer than half of the scores were less than 85, and fewer than half of the scores were greater than 85, the median score is 85. The difference between the mean and the median score on this exam is $86 - 85 = 1$.

20. **(C)** Each digit appears the same number of times in the 1's place, the 10's place, ..., and the 10,000's place. The average of the digits in each place is
$$\frac{1}{5}(1 + 3 + 5 + 7 + 8) = \frac{24}{5} = 4.8.$$
Hence the average of all the numbers is
$$4.8(1 + 10 + 100 + 1000 + 10000) = 4.8(11111) = 53332.8.$$

21. **(A)** The total number of ways that the numbers can be chosen is $\binom{40}{4}$. Exactly 10 of these possibilities result in the four slips having the same number.

Now we need to determine the number of ways that two slips can have a number a and the other two slips have a number b, with $b \neq a$. There are $\binom{10}{2}$ ways to choose the distinct numbers a and b. For each value of a there are $\binom{4}{2}$ to choose the two slips with a and for each value of b there are $\binom{4}{2}$ to choose the two slips with b. Hence the number of ways that two slips have some number a and the other two slips have some distinct number b is

$$\binom{10}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} = 45 \cdot 6 \cdot 6 = 1620.$$

So the probabilities q and p are $\dfrac{10}{\binom{40}{4}}$ and $\dfrac{1620}{\binom{40}{4}}$, respectively, which implies that

$$\frac{p}{q} = \frac{1620}{10} = 162.$$

22. **(C)** Since

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

the condition is equivalent to having an integer value for

$$\frac{n!}{n(n+1)/2}.$$

This reduces, when $n \geq 1$, to having an integer value for

$$\frac{2(n-1)!}{n+1}.$$

This fraction is an integer unless $n+1$ is an odd prime. There are 8 odd primes less than or equal to 25, so there are $24 - 8 = 16$ numbers less than or equal to 24 that satisfy the condition.

23. **(C)** First note that $FE = (AB + DC)/2$. Because trapezoids $ABEF$ and $FECD$ have the same height, the ratio of their areas is equal to the ratio of the averages of their parallel sides. Since

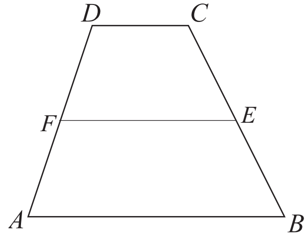
$$AB + \frac{AB + DC}{2} = \frac{3AB + DC}{2}$$

and

$$\frac{AB + DC}{2} + DC = \frac{AB + 3DC}{2},$$

we have

$$3AB + DC = 2(AB + 3DC) = 2AB + 6DC, \quad \text{and} \quad \frac{AB}{DC} = 5.$$



24. **(E)** By the given conditions, it follows that $x > y$. Let $x = 10a + b$ and $y = 10b + a$, where $a > b$. Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since $99(a^2 - b^2)$ must be a perfect square,

$$a^2 - b^2 = (a + b)(a - b) = 11k^2,$$

for some positive integer k. Because a and b are distinct digits, we have $a - b \leq 9 - 1 = 8$ and $a + b \leq 9 + 8 = 17$. It follows that $a + b = 11$, $a - b = k^2$, and k is either 1 or 2.

If $k = 2$, then $(a, b) = (15/2, 7/2)$, which is impossible. Thus $k = 1$ and $(a, b) = (6, 5)$. This gives $x = 65$, $y = 56$, $m = 33$, and $x + y + m = 154$.

25. **(C)** Several pairs of numbers from 1 to 100 sum to 125. These pairs are $(25, 100)$, $(26, 99), \ldots, (62, 63)$. Set B can have at most one number from each of these $62 - 25 + 1 = 38$ pairs. In addition, B can contain all of the numbers $1, 2, \ldots, 24$ since these cannot be paired with any of the available numbers to sum to 125. So B has at most $38 + 24 = 62$ numbers. The set containing the first 62 positive integers, for example, is one of these maximum sets.

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