Tuesday, JANUARY 31, 2006

57th Annual American Mathematics Contest 10

AMC 10

Contest A

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 2.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score in the top 1% on this AMC 10 will be invited to take the 23rd annual American Invitational Mathematics Examination (AIME) on Tuesday, March 7, 2006 or Wednesday, March 22, 2006. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

The publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.

1. Sandwiches at Joe's Fast Food cost \$3 each and sodas cost \$2 each. How many dollars will it cost to purchase 5 sandwiches and 8 sodas?

   (A) 31 **(B)** 32 **(C)** 33 **(D)** 34 **(E)** 35

2. Define $x \otimes y = x^3 - y$. What is $h \otimes (h \otimes h)$?

   (A) $-h$ **(B)** 0 **(C)** h **(D)** $2h$ **(E)** h^3

3. The ratio of Mary's age to Alice's age is $3 : 5$. Alice is 30 years old. How many years old is Mary?

   (A) 15 **(B)** 18 **(C)** 20 **(D)** 24 **(E)** 50

4. A digital watch displays hours and minutes with AM and PM. What is the largest possible sum of the digits in the display?

   (A) 17 **(B)** 19 **(C)** 21 **(D)** 22 **(E)** 23

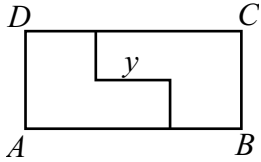
5. Doug and Dave shared a pizza with 8 equally-sized slices. Doug wanted a plain pizza, but Dave wanted anchovies on half of the pizza. The cost of a plain pizza was \$8, and there was an additional cost of \$2 for putting anchovies on one half. Dave ate all the slices of anchovy pizza and one plain slice. Doug ate the remainder. Each then paid for what he had eaten. How many more dollars did Dave pay than Doug?

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

6. What non-zero real value for x satisfies $(7x)^{14} = (14x)^7$?

   (A) $\frac{1}{7}$ **(B)** $\frac{2}{7}$ **(C)** 1 **(D)** 7 **(E)** 14

7. The 8×18 rectangle $ABCD$ is cut into two congruent hexagons, as shown, in such a way that the two hexagons can be repositioned without overlap to form a square. What is y?



   (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

8. A parabola with equation $y = x^2 + bx + c$ passes through the points $(2, 3)$ and $(4, 3)$. What is c?

   (A) 2 **(B)** 5 **(C)** 7 **(D)** 10 **(E)** 11

9. How many sets of two or more consecutive positive integers have a sum of 15?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

10. For how many real values of x is $\sqrt{120 - \sqrt{x}}$ an integer?

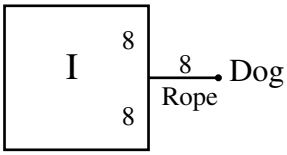
(A) 3 **(B)** 6 **(C)** 9 **(D)** 10 **(E)** 11

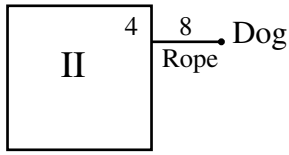
11. Which of the following describes the graph of the equation $(x + y)^2 = x^2 + y^2$?

(A) the empty set **(B)** one point **(C)** two lines **(D)** a circle

(E) the entire plane

12. Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.





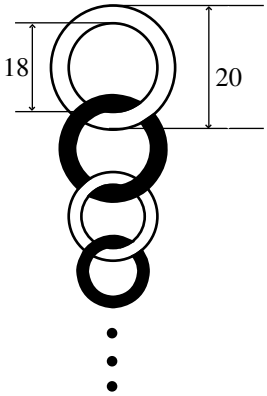
Which of these arrangements gives the dog the greater area to roam, and by how many square feet?

(A) I, by 8π **(B)** I, by 6π **(C)** II, by 4π **(D)** II, by 8π **(E)** II, by 10π

13. A player pays \$5 to play a game. A die is rolled. If the number on the die is odd, the game is lost. If the number on the die is even, the die is rolled again. In this case the player wins if the second number matches the first and loses otherwise. How much should the player win if the game is fair? (In a fair game the probability of winning times the amount won is what the player should pay.)

(A) \$12 **(B)** \$30 **(C)** \$50 **(D)** \$60 **(E)** \$100

14. A number of linked rings, each 1 cm thick, are hanging on a peg. The top ring has an outside diameter of 20 cm. The outside diameter of each of the other rings is 1 cm less than that of the ring above it. The bottom ring has an outside diameter of 3 cm. What is the distance, in cm, from the top of the top ring to the bottom of the bottom ring?

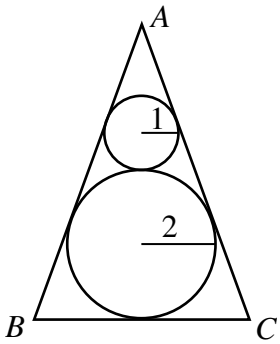


(A) 171 **(B)** 173 **(C)** 182 **(D)** 188 **(E)** 210

15. Odell and Kershaw run for 30 minutes on a circular track. Odell runs clockwise at 250 m/min and uses the inner lane with a radius of 50 meters. Kershaw runs counterclockwise at 300 m/min and uses the outer lane with a radius of 60 meters, starting on the same radial line as Odell. How many times after the start do they pass each other?

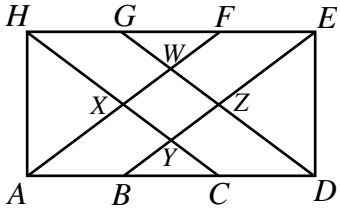
(A) 29 **(B)** 42 **(C)** 45 **(D)** 47 **(E)** 50

16. A circle of radius 1 is tangent to a circle of radius 2. The sides of $\triangle ABC$ are tangent to the circles as shown, and the sides $\overline{AB}$ and $\overline{AC}$ are congruent. What is the area of $\triangle ABC$?



(A) $\frac{35}{2}$ **(B)** $15\sqrt{2}$ **(C)** $\frac{64}{3}$ **(D)** $16\sqrt{2}$ **(E)** 24

17. In rectangle $ADEH$, points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$. In addition, $AH = AC = 2$. What is the area of quadrilateral $WXYZ$ shown in the figure?



(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{2}}{2}$ **(C)** $\frac{\sqrt{3}}{2}$ **(D)** $\frac{2\sqrt{2}}{3}$ **(E)** $\frac{2\sqrt{3}}{3}$

18. A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

(A) $10^4 \cdot 26^2$ **(B)** $10^3 \cdot 26^3$ **(C)** $5 \cdot 10^4 \cdot 26^2$ **(D)** $10^2 \cdot 26^4$ **(E)** $5 \cdot 10^3 \cdot 26^3$

19. How many non-similar triangles have angles whose degree measures are distinct positive integers in arithmetic progression?

(A) 0 **(B)** 1 **(C)** 59 **(D)** 89 **(E)** 178

20. Six distinct positive integers are randomly chosen between 1 and 2006, inclusive. What is the probability that some pair of these integers has a difference that is a multiple of 5?

(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{4}{5}$ **(E)** 1

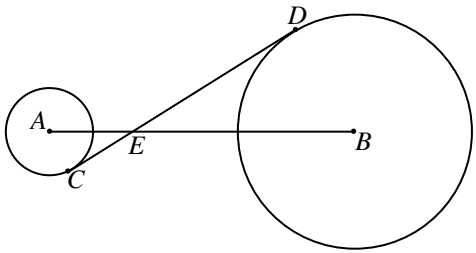
21. How many four-digit positive integers have at least one digit that is a 2 or a 3?

(A) 2439 **(B)** 4096 **(C)** 4903 **(D)** 4904 **(E)** 5416

22. Two farmers agree that pigs are worth \$300 and that goats are worth \$210. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a \$390 debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

(A) \$5 **(B)** \$10 **(C)** \$30 **(D)** \$90 **(E)** \$210

23. Circles with centers A and B have radii 3 and 8, respectively. A common internal tangent touches the circles at C and D, as shown. Lines AB and CD intersect at E, and $AE = 5$. What is CD?



(A) 13 **(B)** $\frac{44}{3}$ **(C)** $\sqrt{221}$ **(D)** $\sqrt{255}$ **(E)** $\frac{55}{3}$

24. Centers of adjacent faces of a unit cube are joined to form a regular octahedron. What is the volume of this octahedron?

(A) $\frac{1}{8}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

25. A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

(A) $\frac{1}{2187}$ **(B)** $\frac{1}{729}$ **(C)** $\frac{2}{243}$ **(D)** $\frac{1}{81}$ **(E)** $\frac{5}{243}$

WRITE TO US!

Correspondence about the problems and solutions for this AMC 10 should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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2006 AIME

The AIME will be held on Tuesday, March 7, 2006 with the alternate on Wednesday, March 22, 2006. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 18 and 19, 2006. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site:

2006

AMC 10 – Contest B

DO NOT OPEN UNTIL WEDNESDAY, February 15, 2006

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE February 15.** Nothing is needed from inside this package until February 15.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*

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