The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

57th Annual American Mathematics Contest 10

AMC 10 – Contest A

Solutions Pamphlet

Tuesday, JANUARY 31, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **(A)** Five sandwiches cost $5 \cdot 3 = 15$ dollars and eight sodas cost $8 \cdot 2 = 16$ dollars. Together they cost $15 + 16 = 31$ dollars.

2. **(C)** By the definition we have

$$h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h.$$

3. **(B)** Mary is $(3/5)(30) = 18$ years old.

4. **(E)** The largest possible sum of the two digits representing the minutes is $5+9 = 14$, occurring at 59 minutes past each hour. The largest possible single digit that can represent the hour is 9. This exceeds the largest possible sum of two digits that can represent the hour, which is $1 + 2 = 3$. Therefore, the largest possible sum of all the digits is $14 + 9 = 23$, occurring at 9:59.

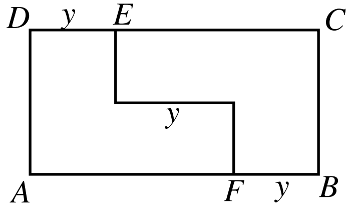
5. **(D)** Each slice of plain pizza cost \$1. Dave paid \$2 for the anchovies in addition to \$5 for the five slices of pizza that he ate, for a total of \$7. Doug paid only \$3 for the three slices of pizza that he ate. Hence Dave paid $7 - 3 = 4$ dollars more than Doug.

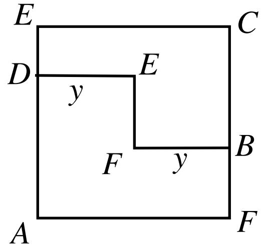
6. **(B)** Take the seventh root of both sides to get $(7x)^2 = 14x$. Then $49x^2 = 14x$, and because $x \neq 0$ we have $49x = 14$. Thus $x = 2/7$.

7. **(A)** Let E represent the end of the cut on $\overline{DC}$, and let F represent the end of the cut on $\overline{AB}$. For a square to be formed, we must have

$$DE = y = FB \quad \text{and} \quad DE + y + FB = 18, \quad \text{so} \quad y = 6.$$

The rectangle that is formed by this cut is indeed a square, since the original rectangle has area $8 \cdot 18 = 144$, and the rectangle that is formed by this cut has a side of length $2 \cdot 6 = 12 = \sqrt{144}$.





8. **(E)** Substitute $(2,3)$ and $(4,3)$ into the equation to give

$$3 = 4 + 2b + c \quad \text{and} \quad 3 = 16 + 4b + c.$$

Subtracting corresponding terms in these equations gives $0 = 12 + 2b$. So

$$b = -6 \quad \text{and} \quad c = 3 - 4 - 2(-6) = 11.$$

OR

The parabola is symmetric about the vertical line through its vertex, and the points $(2,3)$ and $(4,3)$ have the same y-coordinate. The vertex has x-coordinate $(2+4)/2 = 3$, so the equation has the form

$$y = (x-3)^2 + k$$

for some constant k. Since $y = 3$ when $x = 4$, we have $3 = 1^2 + k$ and $k = 2$. Consequently the constant term c is

$$(-3)^2 + k = 9 + 2 = 11.$$

9. **(C)** First note that, in general, the sum of n consecutive integers is n times their median. If the sum is 15, we have the following cases:

if $n = 2$, then the median is 7.5 and the two integers are 7 and 8;

if $n = 3$, then the median is 5 and the three integers are 4, 5, and 6;

if $n = 5$, then the median is 3 and the five integers are 1, 2, 3, 4, and 5.

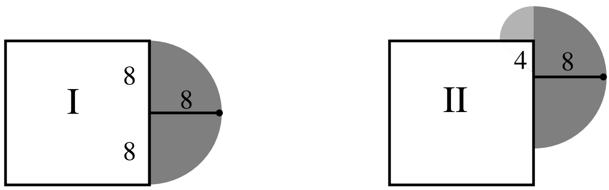
Because the sum of four consecutive integers is even, 15 cannot be written in such a manner. Also, the sum of more than five consecutive integers must be more than $1+2+3+4+5 = 15$. Hence there are 3 sets satisfying the condition.

Note: It can be shown that the number of sets of two or more consecutive positive integers having a sum of k is equal to the number of odd positive divisors of k, excluding 1.

10. **(E)** Suppose that $k = \sqrt{120 - \sqrt{x}}$ is an integer. Then $0 \le k \le \sqrt{120}$, and because k is an integer, we have $0 \le k \le 10$. Thus there are 11 possible integer values of k. For each such k, the corresponding value of x is $\left(120 - k^2\right)^2$. Because $\left(120 - k^2\right)^2$ is positive and decreasing for $0 \le k \le 10$, the 11 values of x are distinct.

11. **(C)** The equation $(x+y)^2 = x^2 + y^2$ is equivalent to $x^2 + 2xy + y^2 = x^2 + y^2$, which reduces to $xy = 0$. Thus the graph of the equation consists of the two lines that are the coordinate axes.

12. **(C)** The regions in which the dog can roam for each arrangement are shaded in the figure. For arrangement I, the area of this region is $\frac{1}{2}\pi \cdot 8^2 = 32\pi$ square feet. The area of the shaded region in arrangement II exceeds this by the area of a quarter-circle of radius 4 feet, that is, by $\frac{1}{4}\pi \cdot 4^2 = 4\pi$ square feet.



13. **(D)** Let x represent the amount the player wins if the game is fair. The chance of an even number is $1/2$, and the chance of matching this number on the second roll is $1/6$. So the probability of winning is $(1/2)(1/6) = 1/12$. Therefore $(1/12)x = \$5$ and $x = \$60$.

14. **(B)** The top of the largest ring is 20 cm above its bottom. That point is 2 cm below the top of the next ring, so it is 17 cm above the bottom of the next ring. The additional distances to the bottoms of the remaining rings are $16\,\text{cm}, 15\,\text{cm}, \ldots, 1\,\text{cm}$. Thus the total distance is

$$20 + (17 + 16 + \cdots + 2 + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 17 \cdot 9 = 173\,\text{cm}.$$

OR

The required distance is the sum of the outside diameters of the 18 rings minus a 2-cm overlap for each of the 17 pairs of consecutive rings. This equals

$$(3+4+5+\cdots+20)-2\cdot 17 = (1+2+3+4+5+\cdots+20)-3-34 = \frac{20 \cdot 21}{2} - 37 = 173\,\text{cm}.$$

15. **(D)** Since Odell's rate is $5/6$ that of Kershaw, but Kershaw's lap distance is $6/5$ that of Odell, they each run a lap in the same time. Hence they pass twice each time they circle the track. Odell runs

$$(30\text{ min})\left(250\frac{\text{m}}{\text{min}}\right)\left(\frac{1}{100\pi}\frac{\text{laps}}{\text{m}}\right) = \frac{75}{\pi}\text{ laps} \approx 23.87\text{ laps},$$

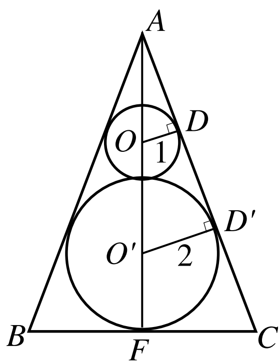
as does Kershaw. Because $23.5 < 23.87 < 24$, they pass each other $2(23.5) = 47$ times.

16. **(D)** Let O and O' denote the centers of the smaller and larger circles, respectively. Let D and D' be the points on $\overline{AC}$ that are also on the smaller and larger circles, respectively. Since $\triangle ADO$ and $\triangle AD'O'$ are similar right triangles, we have

$$\frac{AO}{1} = \frac{AO'}{2} = \frac{AO+3}{2}, \quad \text{so} \quad AO = 3.$$

As a consequence,

$$AD = \sqrt{AO^2 - OD^2} = \sqrt{9-1} = 2\sqrt{2}.$$



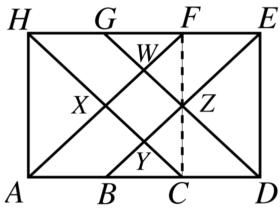
Let F be the midpoint of $\overline{BC}$. Since $\triangle ADO$ and $\triangle AFC$ are similar right triangles, we have

$$\frac{FC}{1} = \frac{AF}{AD} = \frac{AO + OO' + O'F}{AD} = \frac{3+3+2}{2\sqrt{2}} = 2\sqrt{2}.$$

So the area of $\triangle ABC$ is

$$\frac{1}{2} \cdot BC \cdot AF = \frac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}.$$

17. **(A)** First note that since points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$, we have $HG = GF = FE = AB = BC = CD = 1$. Also, $\overline{HG}$ is parallel to $\overline{CD}$ and $HG = CD$, so $CDGH$ is a parallelogram. Similarly, $\overline{AB}$ is parallel to $\overline{FE}$ and $AB = FE$, so $ABEF$ is a parallelogram. As a consequence, $WXYZ$ is a parallelogram, and since $HG = CD = AB = FE$, it is a rhombus.



Since $AH = AC = 2$, the rectangle $ACFH$ is a square of side length 2. Its diagonals $\overline{AF}$ and $\overline{CH}$ have length $2\sqrt{2}$ and form a right angle at X. As a consequence, $WXYZ$ is a square. In isosceles $\triangle HXF$ we have $HX = XF = \sqrt{2}$. In addition, $HG = \frac{1}{2}HF$. So $XW = \frac{1}{2}XF = \frac{1}{2}\sqrt{2}$, and the square $WXYZ$ has area $XW^2 = 1/2$.

18. **(C)** Since the two letters have to be next to each other, think of them as forming a two-letter word w. So each license plate consists of 4 digits and w. For each digit there are 10 choices. There are $26 \cdot 26$ choices for the letters of w, and there are 5 choices for the position of w. So the total number of distinct license plates is $5 \cdot 10^4 \cdot 26^2$.

19. **(C)** Let $n - d$, n, and $n + d$ be the angles in the triangle. Then

$$180 = n - d + n + n + d = 3n, \quad \text{so} \quad n = 60.$$

Because the sum of the degree measures of two angles of a triangle is less than 180, we have

$$180 > n + (n + d) = 120 + d, \quad \text{which implies that } 0 < d < 60.$$

There are 59 triangles with this property.

20. **(E)** Place each of the integers in a pile based on the remainder when the integer is divided by 5. Since there are only 5 piles but there are 6 integers, at least one of the piles must contain two or more integers. The difference of two integers in the same pile is divisible by 5. Hence the probability is 1.

We have applied what is called the Pigeonhole Principle. This states that if you have more pigeons than boxes and you put each pigeon in a box, then at least one of the boxes must have more than one pigeon. In this problem the pigeons are integers and the boxes are piles.

21. **(E)** There are 9000 four-digit positive integers. For those without a 2 or 3, the first digit could be one of the seven numbers 1, 4, 5, 6, 7, 8, or 9, and each of the other digits could be one of the eight numbers 0, 1, 4, 5, 6, 7, 8, or 9. So there are

$$9000 - 7 \cdot 8 \cdot 8 \cdot 8 = 5416$$

four-digit numbers with at least one digit that is a 2 or a 3.

22. **(C)** If a debt of D dollars can be resolved in this way, then integers p and g must exist with

$$D = 300p + 210g = 30(10p + 7g).$$

As a consequence, D must be a multiple of 30, and no positive debt less than \$30 can be resolved. A debt of \$30 can be resolved since

$$30 = 300(-2) + 210(3).$$

This is done by giving 3 goats and receiving 2 pigs.

23. **(B)** Radii $\overline{AC}$ and $\overline{BD}$ are each perpendicular to $\overline{CD}$. By the Pythagorean Theorem,

$$CE = \sqrt{5^2 - 3^2} = 4.$$

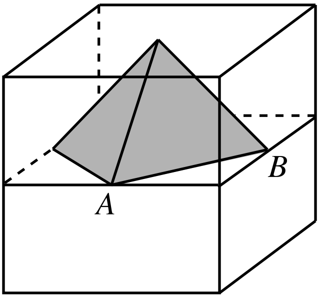
Because $\triangle ACE$ and $\triangle BDE$ are similar,

$$\frac{DE}{CE} = \frac{BD}{AC}, \quad \text{so} \quad DE = CE \cdot \frac{BD}{AC} = 4 \cdot \frac{8}{3} = \frac{32}{3}.$$

Therefore

$$CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$$

24. **(B)** Two pyramids with square bases form the octahedron. The upper pyramid is shown.

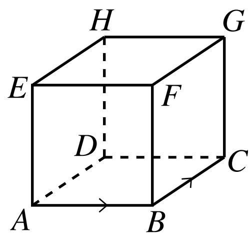


Since the length of $\overline{AB}$ is $\sqrt{2}/2$, the base area of the pyramid is $(\sqrt{2}/2)^2 = 1/2$. The altitude of the pyramid is 1/2, so its volume is

$$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

The volume of the octahedron is $2(1/12) = 1/6$.

25. **(C)** At each vertex there are three possible locations that the bug can travel to in the next move, so the probability that the bug will visit three different vertices after two moves is $2/3$. Label the first three vertices that the bug visits as A to B to C, as shown in the diagram. In order to visit every vertex, the bug must travel from C to either G or D.



The bug travels to G with probability $1/3$, and from there the bug must visit the vertices F, E, H, D in that order. Each of these choices has probability $1/3$ of occurring. So the probability that the path continues in the form

$$C \to G \to F \to E \to H \to D$$

is $\left(\frac{1}{3}\right)^5$.

Alternatively, the bug could travel from C to D with probability $1/3$, and then travel to H, which also occurs with probability $1/3$. From H the bug could go either to G or to E, with probability $2/3$, and from there to the two remaining vertices, each with probability $1/3$. So the probability that the path continues in one of the forms

$$C \to D \to H \begin{array}{l} \nearrow E \to F \to G \\ \searrow G \to F \to E \end{array}$$

is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$.

Hence the bug will visit every vertex in seven moves with probability

$$\left(\frac{2}{3}\right)\left[\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4\right] = \left(\frac{2}{3}\right)\left(\frac{1}{3}+\frac{2}{3}\right)\left(\frac{1}{3}\right)^4 = \frac{2}{243}.$$

OR

From a given starting point there are 3^7 possible walks of seven moves for the bug, all of them equally likely. If such a walk visits every vertex exactly once, there are three choices for the first move and, excluding a return to the start, two choices for the second. Label the first three vertices visited as $A, B,$ and C, in that order, and label the other vertices as shown. The bug must go to either G or D on its third move. In the first case it must then visit vertices $F, E, H,$ and D in order. In the second case it must visit either $H, E, F,$ and G or $H, G, F,$ and E in order. Thus there are $3 \cdot 2 \cdot 3 = 18$ walks that visit every vertex exactly once, so the required probability is $18/3^7 = 2/243$.

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