The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

7th Annual American Mathematics Contest 10

AMC 10 – Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 15, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules.*

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **(C)** Because

$$(-1)^k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ -1, & \text{if } k \text{ is odd,} \end{cases}$$

the sum can be written as

$$(-1+1)+(-1+1)+\cdots+(-1+1) = 0+0+\cdots+0 = 0.$$

2. **(A)** Because $4 \spadesuit 5 = (4+5)(4-5) = -9$, it follows that

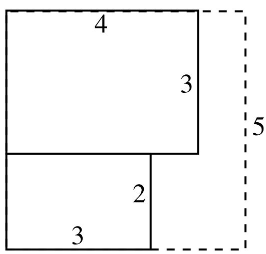
$$3 \spadesuit (4 \spadesuit 5) = 3 \spadesuit (-9) = (3+(-9))(3-(-9)) = (-6)(12) = -72.$$

3. **(A)** Let c and p represent the number of points scored by the Cougars and the Panthers, respectively. The two teams scored a total of 34 points, so $c+p=34$. The Cougars won by 14 points, so $c-p=14$. The solution is $c=24$ and $p=10$, so the Panthers scored 10 points.

4. **(D)** The circle with diameter 3 has area $\pi\left(\frac{3}{2}\right)^2$. The circle with diameter 1 has area $\pi\left(\frac{1}{2}\right)^2$. Therefore the ratio of the blue-painted area to the red-painted area is

$$\frac{\pi\left(\frac{3}{2}\right)^2 - \pi\left(\frac{1}{2}\right)^2}{\pi\left(\frac{1}{2}\right)^2} = 8.$$

5. **(B)** The side length of the square is at least equal to the sum of the smaller dimensions of the rectangles, which is $2+3=5$.



If the rectangles are placed as shown, it is in fact possible to contain them within a square of side length 5. Thus the smallest possible area is $5^2 = 25$.

6. **(D)** Since the square has side length $2/\pi$, the diameter of each circular section is $2/\pi$. The boundary of the region consists of 4 semicircles, whose total perimeter is twice the circumference of a circle having diameter $2/\pi$. Hence the perimeter of the region is

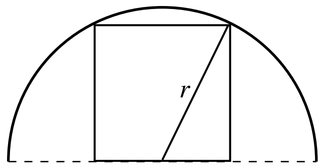
$$2 \cdot \left(\pi \cdot \frac{2}{\pi}\right) = 4.$$

7. **(A)** We have

$$\sqrt{\frac{x}{1-\frac{x-1}{x}}} = \sqrt{\frac{x}{\frac{x-x+1}{x}}} = \sqrt{\frac{x}{\frac{1}{x}}} = \sqrt{x^2} = |x|.$$

When $x < 0$, the given expression is equivalent to $-x$.

8. **(B)**The square has side length $\sqrt{40}$.



Let r be the radius of the semicircle. Then

$$r^2 = \left(\sqrt{40}\right)^2 + \left(\frac{\sqrt{40}}{2}\right)^2 = 40 + 10 = 50,$$

so the area of the semicircle is $\frac{1}{2}\pi r^2 = 25\pi$.

9. **(B)** Francesca's 600 grams of lemonade contains $25 + 386 = 411$ calories, so 200 grams of her lemonade contains $411/3 = 137$ calories.

10. **(A)** Let the sides of the triangle have lengths x, $3x$, and 15. The Triangle Inequality implies that $3x < x + 15$, so $x < 7.5$. Because x is an integer, the greatest possible perimeter is $7 + 21 + 15 = 43$.

11. **(C)** Since $n!$ contains the product $2 \cdot 5 \cdot 10 = 100$ whenever $n \geq 10$, it suffices to determine the tens digit of

$$7! + 8! + 9! = 7!(1 + 8 + 8 \cdot 9) = 5040(1 + 8 + 72) = 5040 \cdot 81.$$

This is the same as the units digit of $4 \cdot 1$, which is 4.

12. **(E)** Substituting $x = 1$ and $y = 2$ into the equations gives

$$1 = \frac{2}{4} + a \quad \text{and} \quad 2 = \frac{1}{4} + b.$$

It follows that

$$a + b = \left(1 - \frac{2}{4}\right) + \left(2 - \frac{1}{4}\right) = 3 - \frac{3}{4} = \frac{9}{4}.$$

OR

Because

$$a = x - \frac{y}{4} \quad \text{and} \quad b = y - \frac{x}{4} \quad \text{we have} \quad a + b = \frac{3}{4}(x + y).$$

Since $x = 1$ when $y = 2$, this implies that $a + b = \frac{3}{4}(1 + 2) = \frac{9}{4}$.

13. **(E)** Joe has 2 ounces of cream in his cup. JoAnn has drunk 2 ounces of the 14 ounces of coffee-cream mixture in her cup, so she has only $12/14 = 6/7$ of her 2 ounces of cream in her cup. Therefore the ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee is

$$\frac{2}{\frac{6}{7}\cdot 2} = \frac{7}{6}.$$

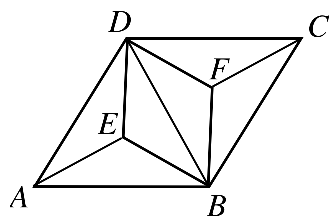
14. **(D)** Since a and b are roots of $x^2 - mx + 2 = 0$, we have

$$x^2 - mx + 2 = (x-a)(x-b) \quad \text{and} \quad ab = 2.$$

In a similar manner, the constant term of $x^2 - px + q$ is the product of $a + (1/b)$ and $b + (1/a)$, so

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = \frac{9}{2}.$$

15. **(C)** Since $\angle BAD = 60^\circ$, isosceles $\triangle BAD$ is also equilateral. As a consequence, $\triangle AEB$, $\triangle AED$, $\triangle BED$, $\triangle BFD$, $\triangle BFC$, and $\triangle CFD$ are congruent. These six triangles have equal areas and their union forms rhombus $ABCD$, so each has area $24/6 = 4$. Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, so its area is 8.

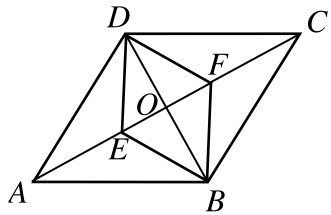


OR

Let the diagonals of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus intersect at right angles, $\triangle ABO$ is a $30-60-90^\circ$ triangle. Therefore $AO = \sqrt{3}\cdot BO$. Because AO and BO are half the length of the longer diagonals of rhombi $ABCD$ and $BFDE$, respectively, it follows that

$$\frac{\text{Area}(BFDE)}{\text{Area}(ABCD)} = \left(\frac{BO}{AO}\right)^2 = \frac{1}{3}.$$

Thus the area of rhombus $BFDE$ is $(1/3)(24) = 8$.



16. **(E)** In the years from 2004 through 2020, Each Leap Day occurs $3 \cdot 365 + 366 = 1461$ days after the preceding Leap Day. When 1461 is divided by 7 the remainder is 5. So the day of the week advances 5 days for each 4-year cycle. In the four cycles from 2004 to 2020, the Leap Day will advance 20 days. So Leap Day in 2020 will occur one day of the week earlier than in 2004, that is, on a Saturday.

17. **(D)** After Alice puts the ball into Bob's bag, his bag will contain six balls: two of one color and one of each of the other colors. After Bob selects a ball and places it into Alice's bag, the two bags will have the same contents if and only if Bob picked one of the two balls in his bag that are the same color. Because there are six balls in the bag when Bob makes his selection, the probability of selecting one of the same colored pair is $2/6 = 1/3$.

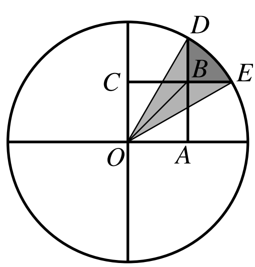
18. **(E)** Note that the first several terms of the sequence are:

$$2,\ 3,\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ 2,\ 3,\ \ldots,$$

so the sequence consists of a repeating cycle of 6 terms. Since $2006 = 334 \cdot 6 + 2$, we have $a_{2006} = a_2 = 3$.

19. **(A)** Since $OC = 1$ and $OE = 2$, it follows that $\angle EOC = 60^\circ$ and $\angle EOA = 30^\circ$. The area of the shaded region is the area of the 30° sector DOE minus the area of congruent triangles OBD and OBE. First note that

$$\text{Area(Sector } DOE) = \frac{1}{12}(4\pi) = \frac{\pi}{3}.$$



In right triangle OCE, we have $CE = \sqrt{3}$, so $BE = \sqrt{3} - 1$. Therefore

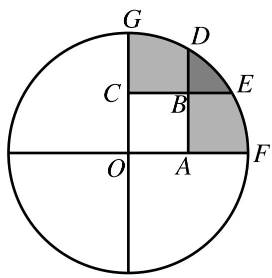
$$\text{Area}(\triangle OBE) = \frac{1}{2}(\sqrt{3} - 1)(1).$$

The required area is consequently

$$\frac{\pi}{3} - 2\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi}{3} + 1 - \sqrt{3}.$$

OR

Let F be the point where ray OA intersects the circle, and let G be the point where ray OC intersects the circle.



Let a be the area of the shaded region described in the problem, and b be the area of the region bounded by $\overline{AD}$, $\overline{AF}$, and the minor arc from D to F. Then b is also the area of the region bounded by $\overline{CE}$, $\overline{CG}$, and the minor arc from G to E. By the Inclusion-Exclusion Principle,

$$2b - a = \text{Area (Quartercircle } OFG) - \text{Area (Square } OABC) = \pi - 1.$$

Since b is the area of a 60° sector from which the area of $\triangle OAD$ has been deleted, we have

$$b = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

Hence the area of the shaded region described in the problem is

$$a = 2b - \pi + 1 = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \pi + 1 = \frac{\pi}{3} + 1 - \sqrt{3}.$$

20. **(E)** The slope of line AB is $(178 - (-22))/(2006 - 6) = 1/10$. Since the line AD is perpendicular to the line AB, its slope is -10. This implies that

$$-10 = \frac{y - (-22)}{8 - 6}, \quad \text{so} \quad y = -10(2) - 22 = -42, \quad \text{and} \quad D = (8, -42).$$

As a consequence,

$$AB = \sqrt{2000^2 + 200^2} = 200\sqrt{101} \quad \text{and} \quad AD = \sqrt{2^2 + 20^2} = 2\sqrt{101}.$$

Thus

$$\text{Area}(ABCD) = AB \cdot AD = 400 \cdot 101 = 40{,}400.$$

21. **(C)** On each die the probability of rolling k, for $1 \le k \le 6$, is

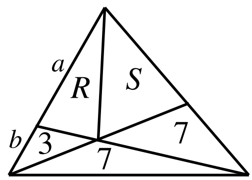
$$\frac{k}{1+2+3+4+5+6} = \frac{k}{21}.$$

There are six ways of rolling a total of 7 on the two dice, represented by the ordered pairs $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Thus the probability of rolling a total of 7 is

$$\frac{1\cdot 6 + 2\cdot 5 + 3\cdot 4 + 4\cdot 3 + 5\cdot 2 + 6\cdot 1}{21^2} = \frac{56}{21^2} = \frac{8}{63}.$$

22. **(D)** The total cost of the peanut butter and jam is $N(4B+5J) = 253$ cents, so N and $4B+5J$ are factors of $253 = 11\cdot 23$. Because $N > 1$, the possible values of N are 11, 23, and 253. If $N = 253$, then $4B+5J = 1$, which is impossible since B and J are positive integers. If $N = 23$, then $4B+5J = 11$, which also has no solutions in positive integers. Hence $N = 11$ and $4B+5J = 23$, which has the unique positive integer solution $B = 2$ and $J = 3$. So the cost of the jam is $11(3)(5¢) = \$1.65$.

23. **(D)** Partition the quadrilateral into two triangles and let the areas of the triangles be R and S as shown. Then the required area is $T = R + S$.



Let a and b, respectively, be the bases of the triangles with areas R and 3, as indicated. If two triangles have the same altitude, then the ratio of their areas is the same as the ratio of their bases. Thus

$$\frac{a}{b} = \frac{R}{3} = \frac{R+S+7}{3+7}, \quad \text{so} \quad \frac{R}{3} = \frac{T+7}{10}.$$

Similarly,

$$\frac{S}{7} = \frac{S+R+3}{7+7}, \quad \text{so} \quad \frac{S}{7} = \frac{T+3}{14}.$$

Thus

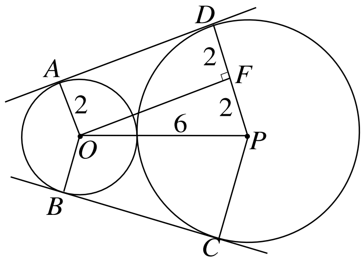
$$T = R + S = 3\left(\frac{T+7}{10}\right) + 7\left(\frac{T+3}{14}\right).$$

From this we obtain

$$10T = 3(T+7) + 5(T+3) = 8T + 36,$$

and it follows that $T = 18$.

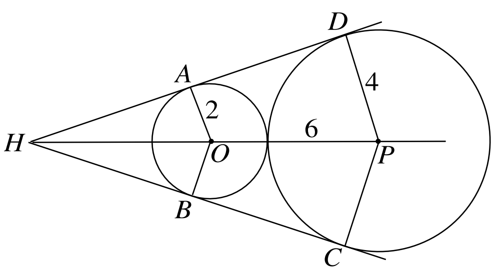
24. **(B)** Through O draw a line parallel to $\overline{AD}$ intersecting $\overline{PD}$ at F.



Then $AOFD$ is a rectangle and OPF is a right triangle. Thus $DF = 2$, $FP = 2$, and $OF = 4\sqrt{2}$. The area of trapezoid $AOPD$ is $12\sqrt{2}$, and the area of hexagon $AOBCPD$ is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

OR

Lines $AD, BC,$ and OP intersect at a common point H.



Because $\angle PDH = \angle OAH = 90^\circ$, triangles PDH and OAH are similar with ratio of similarity 2. Thus $2HO = HP = HO + OP = HO + 6$, so $HO = 6$ and $AH = \sqrt{HO^2 - OA^2} = 4\sqrt{2}$. Hence the area of $\triangle OAH$ is $(1/2)(2)(4\sqrt{2}) = 4\sqrt{2}$, and the area of $\triangle PDH$ is $(2^2)(4\sqrt{2}) = 16\sqrt{2}$. The area of the hexagon is twice the area of $\triangle PDH$ minus twice the area of $\triangle OAH$, so it is $24\sqrt{2}$.

25. **(B)** The 4-digit number on the license plate has the form *aabb* or *abab* or *baab*, where a and b are distinct integers from 0 to 9. Because Mr. Jones has a child of age 9, the number on the license plate is divisible by 9. Hence the sum of the digits, $2(a+b)$, is also divisible by 9. Because of the restriction on the digits a and b, this implies that $a + b = 9$. Moreover, since Mr. Jones must have either a 4-year-old or an 8-year-old, the license plate number is divisible by 4. These conditions narrow the possibilities for the number to 1188, 2772, 3636, 5544, 6336, 7272, and 9900. The last two digits of 9900 could not yield Mr. Jones's age, and none of the others is divisible by 5, so he does not have a 5-year-old.

Note that 5544 is divisible by each of the other eight non-zero digits.

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