Tuesday, FEBRUARY 6, 2007

8th Annual American Mathematics Contest 10

AMC 10
CONTEST A

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, ruler, compass, protractor, erasers and calculators that are accepted for use on the SAT. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form. When your proctor gives the signal, begin working the problems. You will have **75 MINUTES** to complete the test.
8. When you finish the exam, *sign your name* in the space provided on the Answer Form.

Students who score 120 or above or finish in the top 1% on this AMC 10 will be invited to take the 25th annual American Invitational Mathematics Examination (AIME) on Tuesday, March 13, 2007 or Wednesday, March 28, 2007. More details about the AIME and other information are on the back page of this test booklet.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

1. One ticket to a show costs \$20 at full price. Susan buys 4 tickets using a coupon that gives her a 25% discount. Pam buys 5 tickets using a coupon that gives her a 30% discount. How many more dollars does Pam pay than Susan?

   (A) 2 **(B)** 5 **(C)** 10 **(D)** 15 **(E)** 20

2. Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. What is $\dfrac{6@2}{6\#2}$?

   (A) $-\frac{1}{2}$ **(B)** $-\frac{1}{4}$ **(C)** $\frac{1}{8}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

3. An aquarium has a rectangular base that measures 100 cm by 40 cm and has a height of 50 cm. It is filled with water to a height of 40 cm. A brick with a rectangular base that measures 40 cm by 20 cm and a height of 10 cm is placed in the aquarium. By how many centimeters does the water rise?

   (A) 0.5 **(B)** 1 **(C)** 1.5 **(D)** 2 **(E)** 2.5

4. The larger of two consecutive odd integers is three times the smaller. What is their sum?

   (A) 4 **(B)** 8 **(C)** 12 **(D)** 16 **(E)** 20

5. A school store sells 7 pencils and 8 notebooks for \$4.15. It also sells 5 pencils and 3 notebooks for \$1.77. How much do 16 pencils and 10 notebooks cost?

   (A) \$4.76 **(B)** \$5.84 **(C)** \$6.00 **(D)** \$6.16 **(E)** \$6.32

6. At Euclid High School, the number of students taking the AMC10 was 60 in 2002, 66 in 2003, 70 in 2004, 76 in 2005, and 78 in 2006, and is 85 in 2007. Between what two consecutive years was there the largest percentage increase?

   (A) 2002 and 2003 **(B)** 2003 and 2004 **(C)** 2004 and 2005

   (D) 2005 and 2006 **(E)** 2006 and 2007

7. Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of \$10,500 for both taxes. How many dollars was the inheritance?

   (A) 30,000 **(B)** 32,500 **(C)** 35,000 **(D)** 37,500 **(E)** 40,000

8. Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside $\triangle ABC$, $\angle ABC = 40^\circ$, and $\angle ADC = 140^\circ$. What is the degree measure of $\angle BAD$?

   (A) 20 **(B)** 30 **(C)** 40 **(D)** 50 **(E)** 60

9. Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

   (A) -60 **(B)** -17 **(C)** 9 **(D)** 12 **(E)** 60

10. The Dunbar family consists of a mother, a father, and some children. The average age of the members of the family is 20, the father is 48 years old, and the average age of the mother and children is 16. How many children are in the family?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

11. The numbers from 1 to 8 are placed at the vertices of a cube in such a manner that the sum of the four numbers on each face is the same. What is this common sum?

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

12. Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

(A) 56 **(B)** 58 **(C)** 60 **(D)** 62 **(E)** 64

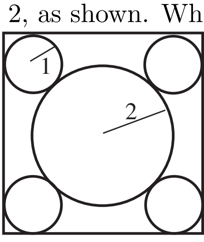
13. Yan is somewhere between his home and the stadium. To get to the stadium he can walk directly to the stadium, or else he can walk home and then ride his bicycle to the stadium. He rides 7 times as fast as he walks, and both choices require the same amount of time. What is the ratio of Yan's distance from his home to his distance from the stadium?

(A) $\frac{2}{3}$ **(B)** $\frac{3}{4}$ **(C)** $\frac{4}{5}$ **(D)** $\frac{5}{6}$ **(E)** $\frac{6}{7}$

14. A triangle with side lengths in the ratio 3:4:5 is inscribed in a circle of radius 3. What is the area of the triangle?

(A) 8.64 **(B)** 12 **(C)** 5π **(D)** 17.28 **(E)** 18

15. Four circles of radius 1 are each tangent to two sides of a square and externally tangent to a circle of radius 2, as shown. What is the area of the square?



(A) 32 **(B)** $22 + 12\sqrt{2}$ **(C)** $16 + 16\sqrt{3}$ **(D)** 48 **(E)** $36 + 16\sqrt{2}$

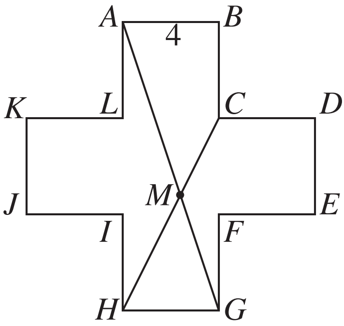
16. Integers a, b, c, and d, not necessarily distinct, are chosen independently and at random from 0 to 2007, inclusive. What is the probability that $ad - bc$ is even?

(A) $\frac{3}{8}$ **(B)** $\frac{7}{16}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{5}{8}$

17. Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

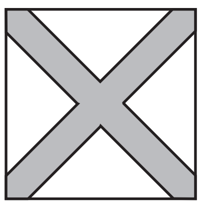
(A) 15 **(B)** 30 **(C)** 50 **(D)** 60 **(E)** 5700

18. Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?



(A) $\frac{44}{3}$ **(B)** 16 **(C)** $\frac{88}{5}$ **(D)** 20 **(E)** $\frac{62}{3}$

19. A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width?



(A) $2\sqrt{2}+1$ **(B)** $3\sqrt{2}$ **(C)** $2\sqrt{2}+2$ **(D)** $3\sqrt{2}+1$ **(E)** $3\sqrt{2}+2$

20. Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

(A) 164 **(B)** 172 **(C)** 192 **(D)** 194 **(E)** 212

21. A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

(A) 3 **(B)** 6 **(C)** 8 **(D)** 9 **(E)** 12

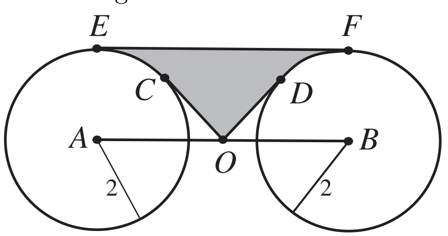
22. A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

(A) 3 **(B)** 7 **(C)** 13 **(D)** 37 **(E)** 43

23. How many ordered pairs (m, n) of positive integers, with $m > n$, have the property that their squares differ by 96?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 12

24. Circles centered at A and B each have radius 2, as shown. Point O is the midpoint of $\overline{AB}$, and $OA = 2\sqrt{2}$. Segments OC and OD are tangent to the circles centered at A and B, respectively, and $\overline{EF}$ is a common tangent. What is the area of the shaded region $ECODF$?



(A) $\dfrac{8\sqrt{2}}{3}$ **(B)** $8\sqrt{2} - 4 - \pi$ **(C)** $4\sqrt{2}$ **(D)** $4\sqrt{2} + \dfrac{\pi}{8}$ **(E)** $8\sqrt{2} - 2 - \dfrac{\pi}{2}$

25. For each positive integer n, let $S(n)$ denote the sum of the digits of n. For how many values of n is $n + S(n) + S(S(n)) = 2007$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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2007 AIME

The AIME will be held on Tuesday, March 13, 2007, with the alternate on Wednesday, March 28, 2007. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above, or finish in the top 1% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the USA Mathematical Olympiad (USAMO) on April 24 and 25, 2007. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: www.unl.edu/amc.

2007

AMC 10 – CONTEST A

DO NOT OPEN UNTIL

TUESDAY, February 6, 2007

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. **PLEASE READ THE MANUAL BEFORE February 6.** Nothing is needed from inside this package until February 6.
2. Your PRINCIPAL or VICE PRINCIPAL must sign the Certification Form found in the Teachers' Manual.
3. The Answer Forms must be mailed by First Class mail to the AMC no later than 24 hours following the examination.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination during this period via copier, telephone, email, World Wide Web or media of any type is a violation of the competition rules.*