THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

8th Annual American Mathematics Contest 10

AMC 10 CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 6, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (C):** Susan pays $(4)(0.75)(20) = 60$ dollars. Pam pays $(5)(0.70)(20) = 70$ dollars, so she pays $70 - 60 = 10$ more dollars than Susan.

2. **Answer (A):** The value of 6@2 is $6 \cdot 2 - 2^2 = 12 - 4 = 8$, and the value of 6#2 is $6 + 2 - 6 \cdot 2^2 = 8 - 24 = -16$. Thus

$$\frac{6@2}{6\#2} = \frac{8}{-16} = -\frac{1}{2}.$$

3. **Answer (D):** The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters. Suppose that after the brick is placed in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have

$$8000 = 4000h \quad \text{and} \quad h = 2 \text{ centimeters}$$

4. **Answer (A):** Let the smaller of the integers be x. Then the larger is $x + 2$. So $x + 2 = 3x$, from which $x = 1$. Thus the two integers are 1 and 3, and their sum is 4.

5. **Answer (B):** Let p be the cost in cents of a pencil and n be the cost in cents of a notebook. Then

$$7p + 8n = 415 \quad \text{and} \quad 5p + 3n = 177.$$

The solution of this pair of equations is $p = 9$ and $n = 44$. So the cost of 16 pencils and 10 notebooks is $16(9) + 10(44) = 584$ cents, or \$5.84.

6. **Answer (A):** Between 2002 and 2003, the increase was

$$\frac{6}{60} = \frac{1}{10} = 10\%.$$

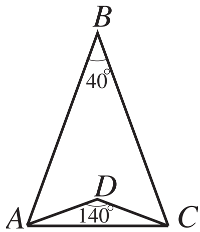
Between the other four pairs of consecutive years, the increases were

$$\frac{4}{66} < \frac{4}{40} = \frac{1}{10}, \quad \frac{6}{70} < \frac{6}{60} = \frac{1}{10}, \quad \frac{2}{76} < \frac{2}{20} = \frac{1}{10}, \quad \text{and} \quad \frac{7}{78} < \frac{7}{70} = \frac{1}{10}.$$

Therefore the largest percentage increase occurred between 2002 and 2003.

7. **Answer (D):** After paying the federal taxes, Mr. Public had 80% of his inheritance money left. He paid 10% of that, or 8% of his inheritance, in state taxes. Hence his total tax bill was 28% of his inheritance, and his inheritance was \$10,500/0.28 = \$37,500.

8. **Answer (D):** Because $\triangle ABC$ is isosceles, $\angle BAC = \dfrac{1}{2}(180° - \angle ABC) = 70°$.



Similarly,

$$\angle DAC = \frac{1}{2}(180° - \angle ADC) = 20°.$$

Thus $\angle BAD = \angle BAC - \angle DAC = 50°$.

OR

Because $\triangle ABC$ and $\triangle ADC$ are isosceles triangles and $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$, applying the Exterior Angle Theorem to $\triangle ABD$ gives $\angle BAD = 70° - 20° = 50°$.

9. **Answer (E):** The given equations are equivalent, respectively, to

$$3^a = 3^{4(b+2)} \quad \text{and} \quad 5^{3b} = 5^{a-3}.$$

Therefore $a = 4(b+2)$ and $3b = a - 3$. The solution of this system is $a = -12$ and $b = -5$, so $ab = 60$.

10. **Answer (E):** Let N represent the number of children in the family and T represent the sum of the ages of all the family members. The average age of the members of the family is 20, and the average age of the members when the 48-year-old father is not included is 16, so

$$20 = \frac{T}{N+2} \quad \text{and} \quad 16 = \frac{T-48}{N+1}.$$

This implies that

$$20N + 40 = T \quad \text{and} \quad 16N + 16 = T - 48,$$

so

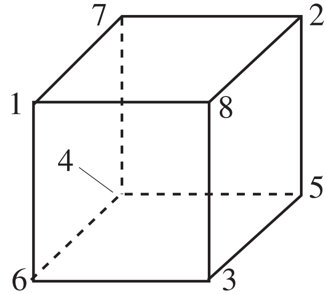
$$20N + 40 = 16N + 64.$$

Hence $4N = 24$ and $N = 6$.

11. **Answer (C):** Each vertex appears on exactly three faces, so the sum of the numbers on all the faces is

$$3(1 + 2 + \cdots + 8) = 3 \cdot \frac{8 \cdot 9}{2} = 108.$$

There are six faces for the cube, so the common sum must be $108/6 = 18$. A possible numbering is shown in the figure.



12. **Answer (D):** The first guide can take any combination of tourists except all the tourists or none of the tourists. Therefore the number of possibilities is

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62.$$

OR

If each guide did not need to take at least one tourist, then each tourist could choose one of the two guides independently. In this case there would be $2^6 = 64$ possible arrangements. The two arrangements for which all tourists choose the same guide must be excluded, leaving a total of $64 - 2 = 62$ possible arrangements.

13. **Answer (B):** Let w be Yan's walking speed, and let x and y be the distances from Yan to his home and to the stadium, respectively. The time required for Yan to walk to the stadium is y/w, and the time required for him to walk home is x/w. Because he rides his bicycle at a speed of $7w$, the time required for him to ride his bicycle from his home to the stadium is $(x + y)/(7w)$. Thus

$$\frac{y}{w} = \frac{x}{w} + \frac{x+y}{7w} = \frac{8x+y}{7w}.$$

As a consequence, $7y = 8x + y$, so $8x = 6y$. The required ratio is $x/y = 6/8 = 3/4$.

OR

Because we are interested only in the ratio of the distances, we may assume that the distance from Yan's home to the stadium is 1 mile. Let x be his present distance from his home. Imagine that Yan has a twin, Nay. While Yan walks to the stadium, Nay walks to their home and continues $1/7$ of a mile past their home. Because walking $1/7$ of a mile requires the same amount of time as riding 1 mile, Yan and Nay will complete their trips at the same time. Yan has walked $1 - x$ miles while Nay has walked $x + \frac{1}{7}$ miles, so $1 - x = x + \frac{1}{7}$. Thus $x = 3/7$, $1 - x = 4/7$, and the required ratio is $x/(1 - x) = 3/4$.

14. **Answer (A):** Let the sides of the triangle have lengths $3x$, $4x$, and $5x$. The triangle is a right triangle, so its hypotenuse is a diameter of the circle. Thus $5x = 2 \cdot 3 = 6$, so $x = 6/5$. The area of the triangle is

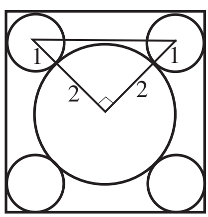
$$\frac{1}{2} \cdot 3x \cdot 4x = \frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

OR

A right triangle with side lengths 3, 4, and 5 has area $(1/2)(3)(4) = 6$. Because the given right triangle is inscribed in a circle with diameter 6, the hypotenuse of this triangle has length 6. Thus the sides of the given triangle are $6/5$ as long as those of a $3-4-5$ triangle, and its area is $(6/5)^2$ times that of a $3-4-5$ triangle. The area of the given triangle is

$$\left(\frac{6}{5}\right)^2 (6) = \frac{216}{25} = 8.64.$$

15. **Answer (B):** Let s be the length of a side of the square. Consider an isosceles right triangle with vertices at the centers of the circle of radius 2 and two of the circles of radius 1. This triangle has legs of length 3, so its hypotenuse has length $3\sqrt{2}$.



The length of a side of the square is 2 more than the length of this hypotenuse, so $s = 2 + 3\sqrt{2}$. Hence the area of the square is

$$s^2 = (2 + 3\sqrt{2})^2 = 22 + 12\sqrt{2}.$$

OR

The distance from a vertex of the square to the center of the nearest small circle is $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the distance between the centers of two small circles in opposite corners of the square is $1 + 4 + 1 = 6$. Therefore each diagonal of the square has length $6 + 2\sqrt{2}$, and each side has length

$$s = \frac{6 + 2\sqrt{2}}{\sqrt{2}} = 2 + 3\sqrt{2}.$$

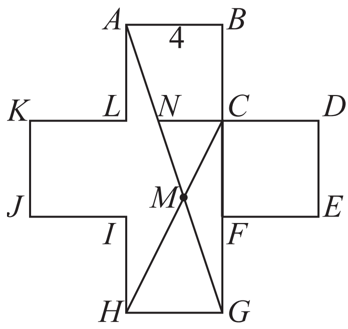
The area of the square is consequently $s^2 = \left(2 + 3\sqrt{2}\right)^2 = 22 + 12\sqrt{2}$.

16. **Answer (E):** The number $ad - bc$ is even if and only if ad and bc are both odd or are both even. Each of ad and bc is odd if both of its factors are odd, and even otherwise. Exactly half of the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $(1/2)\cdot(1/2) = 1/4$ and are even with probability $3/4$. Hence the probability that $ad - bc$ is even is

$$\frac{1}{4}\cdot\frac{1}{4}+\frac{3}{4}\cdot\frac{3}{4}=\frac{5}{8}.$$

17. **Answer (D):** An integer is a cube if and only if, in the prime factorization of the number, each prime factor occurs a multiple of three times. Because $n^3 = 75m = 3\cdot 5^2\cdot m$, the minimum value for m is $3^2\cdot 5 = 45$. In that case $n = 15$, and $m + n = 60$.

18. **Answer (C):** Extend $\overline{CD}$ past C to meet $\overline{AG}$ at N.



Since $\triangle ABG$ is similar to $\triangle NCG$,

$$NC = AB\cdot\frac{CG}{BG} = 4\cdot\frac{8}{12}=\frac{8}{3}.$$

This implies that trapezoid $ABCN$ has area

$$\frac{1}{2}\cdot\left(\frac{8}{3}+4\right)\cdot 4=\frac{40}{3}.$$

Let v denote the length of the perpendicular from M to $\overline{NC}$. Since $\triangle CMN$ is similar to $\triangle HMG$, and

$$\frac{GH}{NC}=\frac{4}{8/3}=\frac{3}{2},$$

the length of the perpendicular from M to $\overline{HG}$ is $\frac{3}{2}v$. Because

$$v+\frac{3}{2}v=8,\quad \text{we have} \quad v=\frac{16}{5}.$$

Hence the area of $\triangle CMN$ is

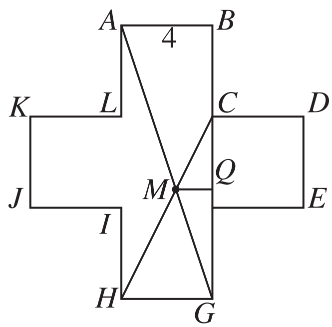
$$\frac{1}{2}\cdot\frac{8}{3}\cdot\frac{16}{5}=\frac{64}{15}.$$

So

$$\text{Area}(ABCM) = \text{Area}(ABCN) + \text{Area}(\triangle CMN) = \frac{40}{3} + \frac{64}{15} = \frac{88}{5}.$$

OR

Let Q be the foot of the perpendicular from M to $\overline{BG}$.



Since $\triangle MQG$ is similar to $\triangle ABG$, we have

$$\frac{MQ}{QG} = \frac{AB}{BG} = \frac{4}{12} = \frac{1}{3}.$$

Also, $\triangle MCQ$ is similar to $\triangle HCG$, so

$$\frac{MQ}{CQ} = \frac{HG}{CG} = \frac{4}{8} = \frac{1}{2}.$$

Thus

$$QG = 3MQ = 3\left(\frac{1}{2}CQ\right) = \frac{3}{2}(8 - QG),$$

which implies that

$$QG = \frac{24}{5} \quad \text{and} \quad MQ = \frac{1}{3}QG = \frac{8}{5}.$$

Hence

$$\text{Area}(ABCM) = \text{Area}(\triangle ABG) - \text{Area}(\triangle CMG) = \frac{1}{2}\cdot 4 \cdot 12 - \frac{1}{2}\cdot 8 \cdot \frac{8}{5} = \frac{88}{5}.$$

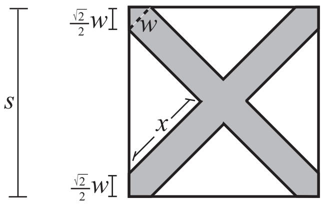
19. **Answer (C):** Let s be the side length of the square, let w be the width of the brush, and let x be the leg length of one of the congruent unpainted isosceles right triangles. Since the unpainted area is half the area of the square, the area of each unpainted triangle is 1/8 of the area of the square. So

$$\frac{1}{2}x^2 = \frac{1}{8}s^2 \quad \text{and} \quad x = \frac{1}{2}s.$$

The leg length x plus the brush width w is equal to half the diagonal of the square, so $x + w = (\sqrt{2}/2)s$. Thus

$$w = \frac{\sqrt{2}}{2}s - \frac{1}{2}s \quad \text{and} \quad \frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

OR



The painted stripes have isosceles right triangles with hypotenuse w at each vertex of the square, and the legs of these triangles have length $(\sqrt{2}/2)w$. Since the total area of the four congruent unpainted triangles is half the area of the original square, we have

$$s - \sqrt{2}w = \frac{s}{\sqrt{2}}, \quad \text{so} \quad \sqrt{2}s - 2w = s.$$

and

$$\frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

20. **Answer (D):** Squaring each side of the equation $4 = a + a^{-1}$ gives

$$16 = a^2 + 2a \cdot a^{-1} + \left(a^{-1}\right)^2 = a^2 + 2 + a^{-2}, \quad \text{so} \quad 14 = a^2 + a^{-2}.$$

Squaring again gives

$$196 = a^4 + 2a^2 \cdot a^{-2} + \left(a^{-2}\right)^2 = a^4 + 2 + a^{-4}, \quad \text{so} \quad 194 = a^4 + a^{-4}.$$

21. **Answer (C):** Since the surface area of the original cube is 24 square meters, each face of the cube has a surface area of $24/6 = 4$ square meters, and the side length of this cube is 2 meters. The sphere inscribed within the cube has diameter 2 meters, which is also the length of the diagonal of the cube inscribed in the sphere. Let l represent the side length of the inscribed cube. Applying the Pythagorean Theorem twice gives

$$l^2 + l^2 + l^2 = 2^2 = 4.$$

Hence each face has surface area

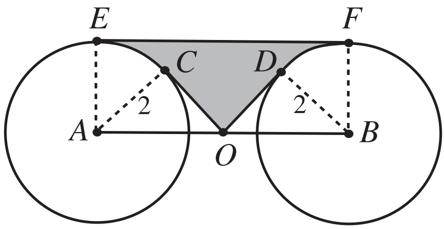
$$l^2 = \frac{4}{3} \text{ square meters.}$$

So the surface area of the inscribed cube is $6 \cdot (4/3) = 8$ square meters.

22. **Answer (D):** A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S = 111k = 3 \cdot 37k$, so S must be divisible by 37. To see that S need not be divisible by any larger prime, note that the sequence 123, 231, 312 gives $S = 666 = 2 \cdot 3^2 \cdot 37$.

23. **Answer (B):** Let x and y be, respectively, the larger and smaller of the integers. Then $96 = x^2 - y^2 = (x+y)(x-y)$. Because 96 is even, x and y are both even or are both odd. In either case $x+y$ and $x-y$ are both even. Hence there are four possibilities for $(x+y, x-y)$, which are $(48, 2)$, $(24, 4)$, $(16, 6)$, and $(12, 8)$. The four corresponding values of (x, y) are $(25, 23)$, $(14, 10)$, $(11, 5)$, and $(10, 2)$.

24. **Answer (B):** Rectangle $ABFE$ has area $AE \cdot AB = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$. Right triangles ACO and BDO each have hypotenuse $2\sqrt{2}$ and one leg of length 2.



Hence they are each isosceles, and each has area $(1/2)\left(2^2\right) = 2$. Angles CAE and DBF are each $45°$, so sectors CAE and DBF each have area

$$\frac{1}{8} \cdot \pi \cdot 2^2 = \frac{\pi}{2}.$$

Thus the area of the shaded region is

$$8\sqrt{2} - 2 \cdot 2 - 2 \cdot \frac{\pi}{2} = 8\sqrt{2} - 4 - \pi.$$

25. **Answer (D):** If $n \le 2007$, then $S(n) \le S(1999) = 28$. If $n \le 28$, then $S(n) \le S(28) = 10$. Therefore if n satisfies the required condition it must also satisfy

$$n \ge 2007 - 28 - 10 = 1969.$$

In addition, $n, S(n),$ and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n),$ and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \le n \le 1999$, we have $n + S(n) \ge 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

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