THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

8th Annual American Mathematics Contest 10

AMC 10

CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 21, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (E):** The perimeter of each bedroom is $2(12+10) = 44$ feet, so the surface to be painted in each bedroom has an area of $44 \cdot 8 - 60 = 292$ square feet. Since there are 3 bedrooms, Isabella must paint $3 \cdot 292 = 876$ square feet.

2. **Answer (E):** Since $3\star 5 = (3+5)5 = 8\cdot 5 = 40$ and $5\star 3 = (5+3)3 = 8\cdot 3 = 24$, we have

$$3\star 5 - 5\star 3 = 40 - 24 = 16.$$

3. **Answer (B):** The student used $120/30 = 4$ gallons on the trip home and $120/20 = 6$ gallons on the trip back to school. So the average gas mileage for the round trip was

$$\frac{240 \text{ miles}}{10 \text{ gallons}} = 24 \text{ miles per gallon.}$$

4. **Answer (D):** Since $OA = OB = OC$, triangles AOB, BOC, and COA are all isosceles. Hence

$$\angle ABC = \angle ABO + \angle OBC = \frac{180^\circ - 140^\circ}{2} + \frac{180^\circ - 120^\circ}{2} = 50^\circ.$$

OR

Since

$$\angle AOC = 360^\circ - 140^\circ - 120^\circ = 100^\circ,$$

the Central Angle Theorem implies that

$$\angle ABC = \frac{1}{2}\angle AOC = 50^\circ.$$

5. **Answer (D):** Let A, B, C, and D represent the following statements about a person in the land.

A : Is an Arog. B : Is a Braf. C : Is a Crup. D : Is a Dramp.

Then the statement in the first sentence of the problem can be expressed as:

$$A \implies B, \qquad C \implies B, \qquad D \implies A \qquad \text{and} \qquad C \implies D.$$

The most we can conclude is that

$$C \implies D \implies A \implies B.$$

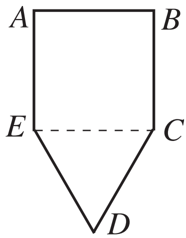
So the only statement listed that we are certain is true is that Crups are both Arogs and Brafs.

6. **Answer (D):** Sarah will receive 4.5 points for the three questions she leaves unanswered, so she must earn at least $100 - 4.5 = 95.5$ points on the first 22 problems. Because

$$15 < \frac{95.5}{6} < 16,$$

she must solve at least 16 of the first 22 problems correctly. This would give her a score of 100.5.

7. **Answer (E):** Because $AB = BC = EA$ and $\angle A = \angle B = 90°$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90°$.



Also $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60°$. Therefore

$$\angle E = \angle AEC + \angle CED = 90° + 60° = 150°.$$

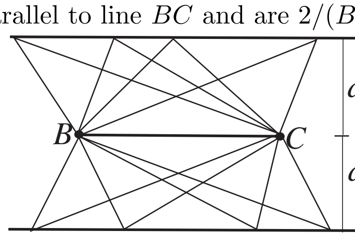
8. **Answer (D):** Once a and c are chosen, the integer b is determined. For $a = 0$, we could have $c = 2$, 4, 6, or 8. For $a = 2$, we could have $c = 4$, 6, or 8. For $a = 4$, we could have $c = 6$ or 8, and for $a = 6$ the only possibility is $c = 8$. Thus there are $1 + 2 + 3 + 4 = 10$ possibilities when a is even. Similarly, there are 10 possibilities when a is odd, so the number of possibilities is 20.

9. **Answer (D):** The last s is the 12th appearance of this letter in the message, so it will be replaced by the letter that is

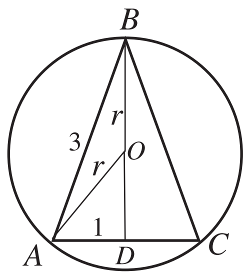
$$1 + 2 + 3 + \cdots + 12 = \frac{1}{2}(12 \cdot 13) = 3 \cdot 26$$

letters to the right of s. Since the alphabet has 26 letters, this letter s is coded as s.

10. **Answer (A):** If the altitude from A has length d, then $\triangle ABC$ has area $(1/2)(BC)d$. The area is 1 if and only if $d = 2/(BC)$. Thus S consists of the two lines that are parallel to line BC and are $2/(BC)$ units from it, as shown.



11. **Answer (C):** Let $\overline{BD}$ be an altitude of the isosceles $\triangle ABC$, and let O denote the center of the circle with radius r that passes through A, B, and C, as shown.



Then

$$BD = \sqrt{3^2 - 1^2} = 2\sqrt{2} \quad \text{and} \quad OD = 2\sqrt{2} - r.$$

Since $\triangle ADO$ is a right triangle, we have

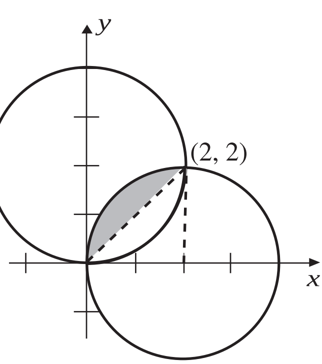
$$r^2 = 1^2 + \left(2\sqrt{2} - r\right)^2 = 1 + 8 - 4\sqrt{2}r + r^2, \quad \text{and} \quad r = \frac{9}{4\sqrt{2}} = \frac{9}{8}\sqrt{2}.$$

As a consequence, the circle has area

$$\left(\frac{9}{8}\sqrt{2}\right)^2 \pi = \frac{81}{32}\pi.$$

12. **Answer (D):** Tom's age N years ago was $T - N$. The sum of his three children's ages at that time was $T - 3N$. Therefore $T - N = 2(T - 3N)$, so $5N = T$ and $T/N = 5$. The conditions of the problem can be met, for example, if Tom's age is 30 and the ages of his children are 9, 10, and 11. In that case $T = 30$ and $N = 6$.

13. **Answer (D):** The two circles intersect at $(0, 0)$ and $(2, 2)$, as shown.



Half of the region described is formed by removing an isosceles right triangle of leg length 2 from a quarter of one of the circles. Because the quarter-circle has area $(1/4)\pi(2)^2 = \pi$ and the triangle has area $(1/2)(2)^2 = 2$, the area of the region is $2(\pi - 2)$.

14. **Answer (C):** Let g be the number of girls and b the number of boys initially in the group. Then $g = 0.4(g + b)$. After two girls leave and two boys arrive, the size of the entire group is unchanged, so $g - 2 = 0.3(g + b)$. The solution of the system of equations

$$g = 0.4(g + b) \quad \text{and} \quad g - 2 = 0.3(g + b)$$

is $g = 8$ and $b = 12$, so there were initially 8 girls.

OR

After two girls leave and two boys arrive, the size of the group is unchanged. So the two girls who left represent $40\% - 30\% = 10\%$ of the group. Thus the size of the group is 20, and the original number of girls was 40% of 20, or 8.

15. **Answer (D):** Let x be the degree measure of $\angle A$. Then the degree measures of angles B, C, and D are $x/2$, $x/3$, and $x/4$, respectively. The degree measures of the four angles have a sum of 360, so

$$360 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12}.$$

Thus $x = (12 \cdot 360)/25 = 172.8 \approx 173$.

16. **Answer (C):** Let N be the number of students in the class. Then there are $0.1N$ juniors and $0.9N$ seniors. Let s be the score of each junior. The scores totaled $84N = 83(0.9N) + s(0.1N)$, so

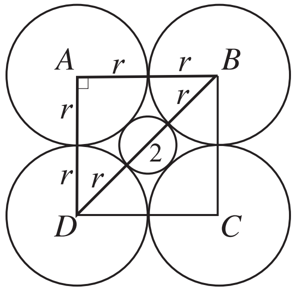
$$s = \frac{84N - 83(0.9N)}{0.1N} = 93.$$

Note: In this problem, we could assume that the class has one junior and nine seniors. Then

$$9 \cdot 83 + s = 10 \cdot 84 = 9 \cdot 84 + 84 \quad \text{and} \quad s = 9(84 - 83) + 84 = 93.$$

17. **Answer (D):** Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

18. **Answer (B):** Construct the square $ABCD$ by connecting the centers of the large circles, as shown, and consider the isosceles right $\triangle BAD$.



Since $AB = AD = 2r$ and $BD = 2 + 2r$, we have $2(2r)^2 = (2+2r)^2$. So

$$1 + 2r + r^2 = 2r^2, \quad \text{and} \quad r^2 - 2r - 1 = 0.$$

Applying the quadratic formula gives $r = 1 + \sqrt{2}$.

19. **Answer (C):** The first remainder is even with probability $2/6 = 1/3$ and odd with probability $2/3$. The second remainder is even with probability $3/6 = 1/2$ and odd with probability $1/2$. The shaded squares are those that indicate that both remainders are odd or both are even. Hence the square is shaded with probability

$$\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}.$$

20. **Answer (C):** After one of the 25 blocks is chosen, 16 of the remaining blocks do not share its row or column. After the second block is chosen, 9 of the remaining blocks do not share a row or column with either of the first two. Because the three blocks can be chosen in any order, the number of different combinations is

$$\frac{25 \cdot 16 \cdot 9}{3!} = 25 \cdot 8 \cdot 3 = 600.$$

21. **Answer (B):** Let s be the side length of the square, and let h be the length of the altitude of $\triangle ABC$ from B. Because $\triangle ABC$ and $\triangle WBZ$ are similar, it follows that

$$\frac{h-s}{s} = \frac{h}{AC} = \frac{h}{5}, \quad \text{so} \quad s = \frac{5h}{5+h}.$$

Because $h = 3 \cdot 4/5 = 12/5$, the side length of the square is

$$s = \frac{5(12/5)}{5 + 12/5} = \frac{60}{37}.$$

OR

Because $\triangle WBZ$ is similar to $\triangle ABC$, we have

$$BZ = \frac{4}{5}s \quad \text{and} \quad CZ = 4 - \frac{4}{5}s.$$

Because $\triangle ZYC$ is similar to $\triangle ABC$, we have

$$\frac{s}{4-(4/5)s} = \frac{3}{5}.$$

Thus

$$5s = 12 - \frac{12}{5}s \quad \text{and} \quad s = \frac{60}{37}.$$

22. **Answer (B):** The probability of the number appearing 0, 1, and 2 times is

$$P(0) = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}, \quad P(1) = 2\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{6}{16}, \quad \text{and} \quad P(2) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16},$$

respectively. So the expected return, in dollars, to the player is

$$P(0)\cdot(-1) + P(1)\cdot(1) + P(2)\cdot(2) = \frac{-9+6+2}{16} = -\frac{1}{16}.$$

23. **Answer (E):** Let h be the altitude of the original pyramid. Then the altitude of the smaller pyramid is $h-2$. Because the two pyramids are similar, the ratio of their altitudes is the square root of the ratio of their surface areas. Thus $h/(h-2) = \sqrt{2}$, so

$$h = \frac{2\sqrt{2}}{\sqrt{2}-1} = 4 + 2\sqrt{2}.$$

24. **Answer (C):** Since n is divisible by 9, the sum of the digits of n must be a multiple of 9. At least one digit of n is 4, so at least nine digits must be 4, and at least one digit must be 9. For n to be divisible by 4, the last two digits of n must each be 4. These conditions are satisfied by several ten-digit numbers, of which the smallest is 4,444,444,944.

25. **Answer (A):** Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that

$$u + \frac{14}{9u} = k$$

for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

OR

Rewrite the equation

$$\frac{a}{b} + \frac{14b}{9a} = k$$

in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

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