THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

9th Annual American Mathematics Contest 10

AMC 10 CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 27, 2008

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Chair: LeRoy Wenstrom, Columbus, MS

1. **Answer (E):** The number of points could be any integer between $5 \cdot 2 = 10$ and $5 \cdot 3 = 15$, inclusive. The number of possibilities is $15 - 10 + 1 = 6$.

2. **Answer (B):** The two sums are $1 + 10 + 17 + 22 = 50$ and $4+9+16+25 = 54$, so the positive difference between the sums is $54 - 50 = 4$.

1	2	3	4
11	10	9	8
15	16	17	18
25	24	23	22

   Query: If a different 4×4 block of dates had been chosen, the answer would be unchanged. Why?

3. **Answer (D):** The properties of exponents imply that

$$\sqrt[3]{x\sqrt{x}} = \left(x \cdot x^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} = x^{\frac{1}{2}}.$$

4. **Answer (C):** A single player can receive the largest possible salary only when the other 20 players on the team are each receiving the minimum salary of \$15,000. Thus the maximum salary for any player is \$700,000 − 20 · \$15,000 = \$400,000.

5. **Answer (A):** Note that $(y - x)^2 = (x - y)^2$, so

$$(x-y)^2\$(y-x)^2 = (x-y)^2\$(x-y)^2 = \left((x-y)^2 - (x-y)^2\right)^2 = 0^2 = 0.$$

6. **Answer (C):** Because $AB + BD = AD$ and $AB = 4BD$, it follows that $BD = \frac{1}{5} \cdot AD$. By similar reasoning, $CD = \frac{1}{10} \cdot AD$. Thus

$$BC = BD - CD = \frac{1}{5} \cdot AD - \frac{1}{10} \cdot AD = \frac{1}{10} \cdot AD.$$

7. **Answer (C):** The side length of the large triangle is 10 times the side length of each small triangle, so the area of the large triangle is $10^2 = 100$ times the area of each small triangle.

8. **Answer (C):** The total cost of the carnations must be an even number of dollars. The total number of dollars spent is the even number 50, so the number of roses purchased must also be even. In addition, the number of roses purchased cannot exceed $\frac{50}{3}$. Therefore the number of roses purchased must be one of the even integers between 0 and 16, inclusive. This gives 9 possibilities for the number of roses purchased, and consequently 9 possibilities for the number of bouquets.

9. **Answer (A):** The quadratic formula implies that the two solutions are

$$x_1 = \frac{2a + \sqrt{4a^2 - 4ab}}{2a} \quad \text{and} \quad x_2 = \frac{2a - \sqrt{4a^2 - 4ab}}{2a},$$

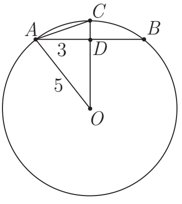
so the average is

$$\frac{1}{2}(x_1 + x_2) = \frac{1}{2}\left(\frac{2a}{2a} + \frac{2a}{2a}\right) = 1.$$

OR

The sum of the solutions of a quadratic equation is the negative of the coefficient of the linear term divided by the coefficient of the quadratic term. In this case the sum of the solution is $\frac{-(-2a)}{a} = 2$. Hence the average of the solutions is 1 .

10. **Answer (A):** Let O be the center of the circle, and let D be the intersection of $\overline{OC}$ and $\overline{AB}$. Because $\overline{OC}$ bisects minor arc AB, $\overline{OD}$ is a perpendicular bisector of chord $\overline{AB}$. Hence $AD = 3$, and applying the Pythagorean Theorem to $\triangle ADO$ yields $OD = \sqrt{5^2 - 3^3} = 4$. Therefore $DC = 1$, and applying the Pythagorean Theorem to $\triangle ADC$ yields $AC = \sqrt{3^2 + 1^2} = \sqrt{10}$.



11. **Answer (B):** Note that $u_5 = 2u_4 + 9$ and $128 = u_6 = 2u_5 + u_4 = 5u_4 + 18$. Thus $u_4 = 22$, and it follows that $u_5 = 2 \cdot 22 + 9 = 53$.

12. **Answer (A):** During the year Pete takes

$$44 \times 10^5 + 5 \times 10^4 = 44.5 \times 10^5$$

steps. At 1800 steps per mile, the number of miles Pete walks is

$$\frac{44.5 \times 10^5}{18 \times 10^2} = \frac{44.5}{18} \times 10^3 \approx 2.5 \times 10^3 = 2500.$$

13. **Answer (B):** Because the mean of the first n terms is n, their sum is n^2. Therefore the nth term is $n^2 - (n-1)^2 = 2n - 1$, and the 2008th term is $2 \cdot 2008 - 1 = 4015$.

14. **Answer (B):** Because $\triangle OAB$ is a $30-60-90°$ triangle, we have $BA = \frac{5\sqrt{3}}{3}$. Let A' and B' be the images of A and B, respectively, under the rotation. Then

$B' = (0, 5)$, $\overline{B'A'}$ is horizontal, and $B'A' = BA = \frac{5\sqrt{3}}{3}$. Hence A' is in the second quadrant and

$$A' = \left(-\frac{5}{3}\sqrt{3}, 5\right).$$

15. **Answer (A):** By the Pythagorean Theorem we have $a^2 + b^2 = (b+1)^2$, so

$$a^2 = (b+1)^2 - b^2 = 2b + 1.$$

Because b is an integer with $b < 100$, a^2 is an odd perfect square between 1 and 201, and there are six of these, namely, 9, 25, 49, 81, 121, and 169. Hence a must be 3, 5, 7, 9, 11, or 13, and there are 6 triangles that satisfy the given conditions.

16. **Answer (A):** If one die is rolled, 3 of the 6 possible numbers are odd. If two dice are rolled, 18 of the 36 possible outcomes have odd sums. In each of these cases, the probability of an odd sum is $\frac{1}{2}$. If no die is rolled, the sum is 0, which is not odd. The probability that no die is rolled is equal to the probability that both coin tosses are tails, which is $(\frac{1}{2})^2 = \frac{1}{4}$. Thus the requested probability is

$$\left(1 - \frac{1}{4}\right) \cdot \frac{1}{2} = \frac{3}{8}.$$

17. **Answer (B):** The responses on these three occasions, in order, must be YNN, NYN, or NNY, where Y indicates approval and N indicates disapproval. The probability of each of these is $(0.7)(0.3)(0.3) = 0.063$, so the requested probability is $3(0.063) = 0.189$.

18. **Answer (B):** Let n be the number of bricks in the chimney. Then the number of bricks per hour Brenda and Brandon can lay working alone is $\frac{n}{9}$ and $\frac{n}{10}$, respectively. Working together they can lay $(\frac{n}{9} + \frac{n}{10} - 10)$ bricks in an hour, or

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right)$$

bricks in 5 hours to complete the chimney. Thus

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right) = n,$$

and the number of bricks in the chimney is $n = 900$.

OR

Suppose that Brenda can lay x bricks in an hour and Brandon can lay y bricks in an hour. Then the number of bricks in the chimney can be expressed as $9x$,

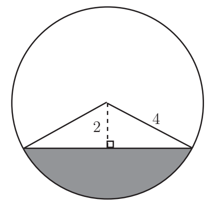
$10y$, or $5(x+y-10)$. The equality of these expressions leads to the system of equations

$$4x - 5y = -50$$

$$-5x + 5y = -50.$$

It follows that $x = 100$, so the number of bricks in the chimney is $9x = 900$.

19. **Answer (E):** The portion of each end of the tank that is under water is a circular sector with two right triangles removed as shown. The hypotenuse of each triangle is 4, and the vertical leg is 2, so each is a $30-60-90°$ triangle. Therefore the sector has a central angle of $120°$, and the area of the sector is



$$\frac{120}{360} \cdot \pi(4)^2 = \frac{16}{3}\pi.$$

The area of each triangle is $\frac{1}{2}(2)\left(2\sqrt{3}\right)$, so the portion of each end that is underwater has area $\frac{16}{3}\pi - 4\sqrt{3}$. The length of the cylinder is 9, so the volume of the water is

$$9\left(\frac{16}{3}\pi - 4\sqrt{3}\right) = 48\pi - 36\sqrt{3}.$$

20. **Answer (B):** Of the 36 possible outcomes, the four pairs $(1,4)$, $(2,3)$, $(2,3)$, and $(4,1)$ yield a sum of 5. The six pairs $(1,6)$, $(2,5)$, $(2,5)$, $(3,4)$, $(3,4)$, and $(4,3)$ yield a sum of 7. The four pairs $(1,8)$, $(3,6)$, $(3,6)$, and $(4,5)$ yield a sum of 9. Thus the probability of getting a sum of 5, 7, or 9 is $(4+6+4)/36 = 7/18$.

    Note: The dice described here are known as Sicherman dice. The probability of obtaining each sum between 2 and 12 is the same as that on a pair of standard dice.

21. **Answer (C):** Let the women be seated first. The first woman may sit in any of the 10 chairs. Because men and women must alternate, the number of choices for the remaining women is 4, 3, 2, and 1. Thus the number of possible seating arrangements for the women is $10 \cdot 4! = 240$. Without loss of generality, suppose that a woman sits in chair 1. Then this woman's spouse must sit in chair 4 or chair 8. If he sits in chair 4 then the women sitting in chairs 7, 3, 9, and 5 must have their spouses sitting in chairs 10, 6, 2, and 8, respectively. If he sits in chair 8 then the women sitting in chairs 5, 9, 3, and 7 must have their spouses sitting in chairs 2, 6, 10, and 4, respectively. So for each possible seating arrangement for the women there are two arrangements for the men. Hence, there are $2 \cdot 240 = 480$ possible seating arrangements.

22. **Answer (C):** There are $6!/(3!2!1!) = 60$ distinguishable orders of the beads on the line. To meet the required condition, the red beads must be placed in

one of four configurations: positions 1, 3, and 5, positions 2, 4, and 6, positions 1, 3, and 6, or positions 1, 4, and 6. In the first two cases, the blue bead can be placed in any of the three remaining positions. In the last two cases, the blue bead can be placed in either of the two adjacent remaining positions. In each case, the placement of the white beads is then determined. Hence there are $2 \cdot 3 + 2 \cdot 2 = 10$ orders that meet the required condition, and the requested probability is $\frac{10}{60} = \frac{1}{6}$.

23. **Answer (B):** Because the area of the border is half the area of the floor, the same is true of the painted rectangle. The painted rectangle measures $a - 2$ by $b - 2$ feet. Hence $ab = 2(a - 2)(b - 2)$, from which $0 = ab - 4a - 4b + 8$. Add 8 to each side of the equation to produce

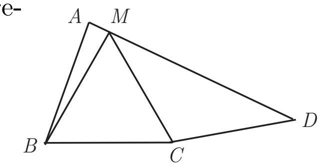
$$8 = ab - 4a - 4b + 16 = (a - 4)(b - 4).$$

Because the only integer factorizations of 8 are

$$8 = 1 \cdot 8 = 2 \cdot 4 = (-4) \cdot (-2) = (-8) \cdot (-1),$$

and because $b > a > 0$, the only possible ordered pairs satisfying this equation for $(a-4, b-4)$ are $(1, 8)$ and $(2, 4)$. Hence (a, b) must be one of the two ordered pairs $(5, 12)$, or $(6, 8)$.

24. **Answer (C):** Let M be on the same side of line BC as A, such that $\triangle BMC$ is equilateral. Then $\triangle ABM$ and $\triangle MCD$ are isosceles with $\angle ABM = 10°$ and $\angle MCD = 110°$. Hence $\angle AMB = 85°$ and $\angle CMD = 35°$. Therefore



$$\begin{aligned}\angle AMD &= 360° - \angle AMB - \angle BMC - \angle CMD \\ &= 360° - 85° - 60° - 35° = 180°.\end{aligned}$$

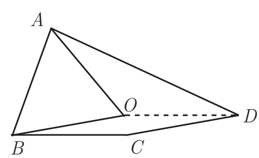
It follows that M lies on $\overline{AD}$ and $\angle BAD = \angle BAM = 85°$.

OR

Let $\triangle ABO$ be equilateral as shown.

Then

$$\angle OBC = \angle ABC - \angle ABO = 70° - 60° = 10°.$$



Because $\angle BCD = 170°$ and $OB = BC = CD$, the quadrilateral $BCDO$ is a parallelogram. Thus

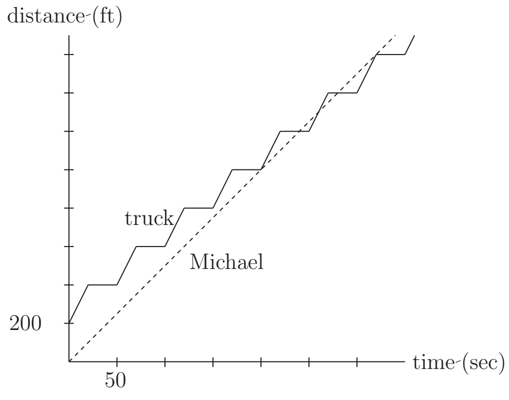
$OD = BC = AO$ and $\triangle AOD$ is isosceles. Let $\alpha = \angle ODA = \angle OAD$. The sum of the interior angles of $ABCD$ is $360°$, so we have

$$360 = (\alpha + 60) + 70 + 170 + (\alpha + 10) \quad \text{and} \quad \alpha = 25.$$

Thus $\angle DAB = 60 + \alpha = 85°$.

25. **Answer (B):** Number the pails consecutively so that Michael is presently at pail 0 and the garbage truck is at pail 1. Michael takes $200/5 = 40$ seconds to walk between pails, so for $n \geq 0$ he passes pail n after $40n$ seconds. The truck takes 20 seconds to travel between pails and stops for 30 seconds at each pail. Thus for $n \geq 1$ it leaves pail n after $50(n-1)$ seconds, and for $n \geq 2$ it arrives at pail n after $50(n-1) - 30$ seconds. Michael will meet the truck at pail n if and only if

$$50(n-1) - 30 \leq 40n \leq 50(n-1) \quad \text{or, equivalently, } 5 \leq n \leq 8.$$



Hence Michael first meets the truck at pail 5 after 200 seconds, just as the truck leaves the pail. He passes the truck at pail 6 after 240 seconds and at pail 7 after 280 seconds. Finally, Michael meets the truck just as it arrives at pail 8 after 320 seconds. These conditions imply that the truck is ahead of Michael between pails 5 and 6 and that Michael is ahead of the truck between pails 7 and 8. However, the truck must pass Michael at some point between pails 6 and 7, so they meet a total of five times.

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