THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

10th Annual American Mathematics Contest 10

AMC 10
CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 25, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (B):** Make a table for the cost of the muffins and bagels:

Cost of Muffins	Cost of Bagels	Total Cost
$0 \cdot 0.50 = 0.00$	$5 \cdot 0.75 = 3.75$	3.75
$1 \cdot 0.50 = 0.50$	$4 \cdot 0.75 = 3.00$	3.50
$2 \cdot 0.50 = 1.00$	$3 \cdot 0.75 = 2.25$	3.25
$3 \cdot 0.50 = 1.50$	$2 \cdot 0.75 = 1.50$	3.00
$4 \cdot 0.50 = 2.00$	$1 \cdot 0.75 = 0.75$	2.75
$5 \cdot 0.50 = 2.50$	$0 \cdot 0.75 = 0.00$	2.50

The only combination which is a whole number of dollars is the cost of 3 muffins and 2 bagels.

2. **Answer (C):** The least common multiple of 2, 3, and 4 is 12, and

$$\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{2}-\frac{1}{3}} \cdot \frac{12}{12} = \frac{4-3}{6-4} = \frac{1}{2}.$$

3. **Answer (C):** The loss of 3 cans of paint resulted in 5 fewer rooms being painted, so the ratio of cans of paint to rooms painted is $3:5$. Hence for 25 rooms she would require $\frac{3}{5} \cdot 25 = 15$ cans of paint.

OR

If she used x cans of paint for 25 rooms, then $\frac{x+3}{30} = \frac{x}{25}$. Hence $25x + 75 = 30x$, and $x = 15$.

4. **Answer (C):** Each triangle has leg length $\frac{1}{2} \cdot (25 - 15) = 5$ meters and area $\frac{1}{2} \cdot 5^2 = \frac{25}{2}$ square meters. Thus the flower beds have a total area of 25 square meters. The entire yard has length 25 and width 5, so its area is 125. The fraction of the yard occupied by the flower beds is $\frac{25}{125} = \frac{1}{5}$.

5. **Answer (D):** Twenty percent less than 60 is $\frac{4}{5} \cdot 60 = 48$. One-third more than a number n is $\frac{4}{3}n$. Therefore $\frac{4}{3}n = 48$, and the number is 36.

6. **Answer (D):** The age of each person is a factor of $128 = 2^7$. So the twins could be $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$ years of age and, consequently, Kiana could be $\frac{128}{1^2} = 128$, $\frac{128}{2^2} = 32$, $\frac{128}{4^2} = 8$, or $\frac{128}{8^2} = 2$ years old, respectively. Because Kiana is younger than her brothers, she must be 2 years old. The sum of their ages is $2 + 8 + 8 = 18$.

7. **Answer (C):** The three operations can be performed in any of $3! = 6$ orders. However, if the addition is performed either first or last, then multiplying in either order produces the same result. Thus at most four distinct values can be obtained. It is easily checked that the values of the four expressions

$$(2 \times 3) + (4 \times 5) = 26,$$
$$((2 \times 3 + 4) \times 5) = 50,$$
$$2 \times (3 + (4 \times 5)) = 46,$$
$$2 \times (3 + 4) \times 5 = 70$$

are in fact all distinct.

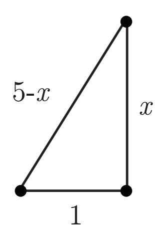
8. **Answer (B):** Let p denote the price at the beginning of January. The price at the end of March was $(1.2)(0.8)(1.25)p = 1.2p$. Because the price at the end of April was p, the price decreased by $0.2p$ during April, and the percent decrease was

$$x = 100 \cdot \frac{0.2p}{1.2p} = \frac{100}{6} \approx 16.7\,.$$

To the nearest integer, x is 17.

9. **Answer (A):** Because $\triangle ABC$ is isosceles, $\angle A = \angle C$. Because $\angle A = \frac{5}{2}\angle B$, we have $\frac{5}{2}\angle B + \frac{5}{2}\angle B + \angle B = 180°$, so $\angle B = 30°$. Therefore $\angle ACB = \angle DCE = 75°$. Because $\triangle CDE$ is isosceles, $2\angle D + 75° = 180°$, so $\angle D = 52.5°$.

10. **Answer (E):** Let x be the height of the stump. Then $5 - x$ is the height of the snapped part, now forming the hypotenuse of a right triangle. By the Pythagorean Theorem,



$$x^2 + 1^2 = (5 - x)^2 = x^2 - 10x + 25$$

from which $x = 2.4$.

11. **Answer (A):** Because the digit 5 appears three times, 5 must be the middle digit of any such palindrome. In the first three digits each of 2, 3, and 5 must appear once and the order in which they appear determines the last three digits. Since there are $3! = 6$ ways to order three distinct digits the number of palindromes is 6.

12. **Answer (A):** The base of the triangle can be 1, 2, or 3, and its altitude is the distance between the two parallel lines, so there are three possible values for the area.

13. **Answer (C):** Define a rotation of the pentagon to be a sequence that starts with $\overline{AB}$ on the x-axis and ends when $\overline{AB}$ is on the x-axis the first time thereafter. Because the pentagon has perimeter 23 and $2009 = 23 \cdot 87 + 8$, it follows that after 87 rotations, point A will be at $x = 23 \cdot 87 = 2001$ and point B will be at $x = 2001 + 3 = 2004$. Points C and D will next touch the x-axis at $x = 2004 + 4 = 2008$ and $x = 2008 + 6 = 2014$, respectively. Therefore a point on $\overline{CD}$ will touch $x = 2009$.

14. **Answer (D):** On Monday, day 1, the birds find $\frac{1}{4}$ quart of millet in the feeder. On Tuesday they find

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4}$$

quarts of millet. On Wednesday, day 3, they find

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

quarts of millet. The number of quarts of millet they find on day n is

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \cdots + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} = \frac{(\frac{1}{4})(1 - (\frac{3}{4})^n)}{1 - \frac{3}{4}} = 1 - \left(\frac{3}{4}\right)^n.$$

The birds always find $\frac{3}{4}$ quart of other seeds, so more than half the seeds are millet if $1 - (\frac{3}{4})^n > \frac{3}{4}$, that is, when $(\frac{3}{4})^n < \frac{1}{4}$. Because $(\frac{3}{4})^4 = \frac{81}{256} > \frac{1}{4}$ and $(\frac{3}{4})^5 = \frac{243}{1024} < \frac{1}{4}$, this will first occur on day 5 which is Friday.

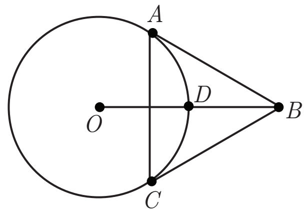
15. **Answer (E):** Let x be the weight of the bucket and let y be the weight of the water in a full bucket. Then we are given that $x + \frac{2}{3}y = a$ and $x + \frac{1}{2}y = b$. Hence $\frac{1}{6}y = a - b$, so $y = 6a - 6b$. Thus $x = b - \frac{1}{2}(6a - 6b) = -3a + 4b$. Finally, $x + y = 3a - 2b$.

OR

The difference between a kg and b kg is the weight of water that would fill $\frac{1}{6}$ of a bucket. So the weight of water that would fill $\frac{1}{2}$ of a bucket is $3(a - b)$. Therefore the weight of a bucket filled with water is $b + 3(a - b) = 3a - 2b$.

16. **Answer (B):** Let the radius of the circle be r. Because $\triangle BCO$ is a right triangle with a $30°$ angle at B, the hypotenuse $\overline{BO}$ is twice as long as $\overline{OC}$, so $BO = 2r$. It follows that $BD = 2r - r = r$, and

$$\frac{BD}{BO} = \frac{r}{2r} = \frac{1}{2}.$$

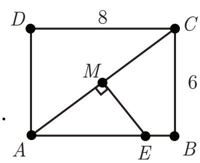


17. **Answer (C):** The area of the entire region is 5. The shaded region consists of a triangle with base $3-a$ and altitude 3, with one unit square removed. Therefore

$$\frac{3(3-a)}{2} - 1 = \frac{5}{2}.$$

Solving this equation yields $a = \frac{2}{3}$.

18. **Answer (D):** By the Pythagorean Theorem, $AC = 10$, so $AM = 5$. Triangles AME and ABC are similar, so $\frac{ME}{AM} = \frac{6}{8}$ and $ME = \frac{15}{4}$. The area of $\triangle AME$ is $\frac{1}{2} \cdot 5 \cdot \frac{15}{4} = \frac{75}{8}$.



OR

As above, $AM = 5$ and $\triangle AME$ and $\triangle ABC$ are similar with similarity ratio 5:8. Therefore

$$Area(\triangle AME) = \left(\frac{5}{8}\right)^2 \cdot Area(\triangle ABC) = \frac{5^2}{8^2} \cdot \frac{8 \cdot 6}{2} = \frac{75}{8}.$$

19. **Answer (A):** The clock will display the incorrect time for the entire hours of 1, 10, 11, and 12. So the correct hour is displayed correctly $\frac{2}{3}$ of the time. The minutes will not display correctly whenever either the tens digit or the ones digit is a 1, so the minutes that will not display correctly are 10, 11, 12, ..., 19, and 01, 21, 31, 41, and 51. This is 15 of the 60 possible minutes for a given hour. Hence the fraction of the day that the clock shows the correct time is $\frac{2}{3} \cdot (1 - \frac{15}{60}) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$.

20. **Answer (B):** By the Pythagorean Theorem, $AC = \sqrt{5}$. By the Angle Bisector Theorem, $\frac{BD}{AB} = \frac{CD}{AC}$. Therefore $CD = \sqrt{5} \cdot BD$ and $BD + CD = 2$, from which

$$BD = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

OR

Let $\overline{DE}$ be an altitude of $\triangle ADC$. Then note that $\triangle ABD$ is congruent to $\triangle AED$, and so $AE = 1$. As in the first solution $AC = \sqrt{5}$. Let $x = BD$. Then $DE = x$, $EC = \sqrt{5} - 1$, and $DC = 2 - x$. Applying the Pythagorean Theorem to $\triangle DEC$ yields $x^2 + \left(\sqrt{5} - 1\right)^2 = (2 - x)^2$, from which $x = \frac{\sqrt{5}-1}{2}$.

21. **Answer (D):** The sum of any four consecutive powers of 3 is divisible by $3^0 + 3^1 + 3^2 + 3^3 = 40$ and hence is divisible by 8. Therefore

$$(3^2 + 3^3 + 3^4 + 3^5) + \cdots + (3^{2006} + 3^{2007} + 3^{2008} + 3^{2009})$$

is divisible by 8. So the required remainder is $3^0 + 3^1 = 4$.

22. **Answer (B):** The area of triangle A is 1, and its hypotenuse has length $\sqrt{5}$. Triangle B is similar to triangle A and has a hypotenuse of 2, so its area is $(\frac{2}{\sqrt{5}})^2 = \frac{4}{5}$. The volume of the required piece is $c = \frac{4}{5} \cdot 2 = \frac{8}{5}$ cubic inches. The icing on this piece has an area of $s = \frac{4}{5} + 2^2 = \frac{24}{5}$ square inches. Therefore $c + s = \frac{8}{5} + \frac{24}{5} = \frac{32}{5}$.

23. **Answer (C):** After 10 min. = 600 sec., Rachel will have completed 6 laps and be 30 seconds from the finish line. Because Rachel runs one-fourth of a lap in 22.5 seconds, she will be in the picture taking region between

$$30 - \frac{22.5}{2} = 18.75 \quad \text{and} \qquad 30 + \frac{22.5}{2} = 41.25$$

seconds of the 10th minute. After 10 minutes Robert will have completed 7 laps and will be 40 seconds from the starting line. Because Robert runs one-fourth of a lap in 20 seconds, he will be in the picture taking region between 30 and 50 seconds of the 10th minute. Hence both Rachel and Robert will be in the picture if it is taken between 30 and 41.25 seconds of the 10th minute. The probability that the picture is snapped during this time is

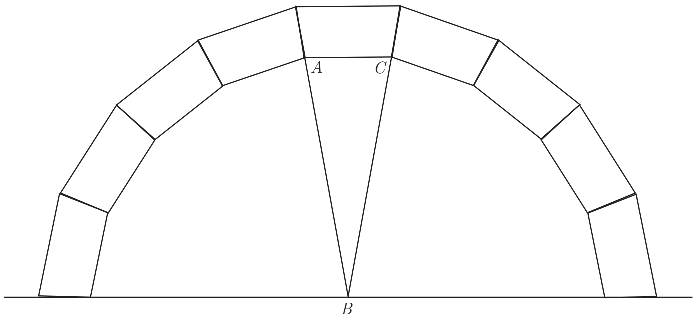
$$\frac{41.25 - 30}{60} = \frac{3}{16}.$$

24. **Answer (A):** Add a symmetric arch to the given arch to create a closed loop of trapezoids. Consider the regular 18-sided polygon created by the interior of the completed loop. Each interior angle of a regular 18-gon measures

$$(18 - 2) \cdot 180^\circ/18 = 160^\circ.$$

Then $x + x + 160^\circ = 360^\circ$, so $x = 100^\circ$.

OR



Extend two sides of a trapezoid until they meet at the center of the arch, as shown. Then $\triangle ABC$ is isosceles and by symmetry $\angle ABC = \frac{180}{9} = 20°$, and $\angle BAC = 80°$. The requested angle is supplementary to $\angle BAC$, so $x = 180 - 80 = 100°$.

25. **Answer (B):** The stripe on each face of the cube will be oriented in one of two possible directions, so there are $2^6 = 64$ possible stripe combinations on the cube. There are 3 pairs of parallel faces so, if there is an encircling stripe, then the pair of faces that do not contribute uniquely determine the stripe orientation for the remaining faces. In addition, the stripe on each face that does not contribute may be oriented in 2 different ways. Thus a total of $3 \cdot 2 \cdot 2 = 12$ stripe combinations on the cube result in a continuous stripe around the cube, and the requested probability is $\frac{12}{64} = \frac{3}{16}$.

OR

Without loss of generality, orient the cube so that the stripe on the top face goes from front to back. There are two mutually exclusive ways for there to be an encircling stripe: either the front, bottom, and back faces are painted to complete an encircling stripe with the top face's stripe, or the front, right, back, and left faces are painted to form an encircling stripe. The probability of the first cases is $(\frac{1}{2})^3 = \frac{1}{8}$, and the probability of the second case is $(\frac{1}{2})^4 = \frac{1}{16}$, so the answer is $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$.

OR

There are three possible orientations of an encircling stripe. For any one of these to appear, the four faces through which the stripe is to pass must be properly aligned. The probability of one such stripe alignment is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Because there are 3 such possibilities, and these events are disjoint, the total probability is $3\left(\frac{1}{16}\right)=\frac{3}{16}$.

The problems and solutions in this contest were proposed by Steve Blasberg, Thomas Butts, Gerald Bergum, Steve Dunbar, Douglas Faires, Sister Josanne Furey, Gregory Galperin, Jerrold Grossman, Elgin Johnston, Joe Kennedy, David Wells, LeRoy Wenstrom, Woody Wenstrom.

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