Mathematical Association of America
American Mathematics Competitions

11th Annual

AMC 10 A

American Mathematics Contest 10A

Solutions Pamphlet

Tuesday, February 9, 2010

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (D):** The average of the five values is

$$\frac{6 + 0.5 + 1 + 2.5 + 10}{5} = \frac{20}{5} = 4.$$

2. **Answer (B):** Let s be the side length of the smaller square. Then the length of the rectangle is $4s$, and the width is $4s - s = 3s$. Hence the rectangle length is $\frac{4s}{3s} = \frac{4}{3}$ times as large as its width.

3. **Answer (D):** Let x be the number of marbles that Tyrone gave to Eric. Then $97 - x = 2(11 + x)$. Solving this equation yields $x = 25$.

4. **Answer (B):** Because $412 \div 56$ is between 7 and 8, the reading will need 8 discs. Therefore each disc will contain $412 \div 8 = 51.5$ minutes of reading.

5. **Answer (E):** Because the circumference is $2\pi r = 24\pi$, the radius r is 12. Therefore the area is $\pi r^2 = 144\pi$, and $k = 144$.

6. **Answer (C):** Note that $\spadesuit(2, 2) = 2 - \frac{1}{2} = \frac{3}{2}$. Therefore

$$\spadesuit(2, \spadesuit(2, 2)) = \spadesuit\left(2, \frac{3}{2}\right) = 2 - \frac{2}{3} = \frac{4}{3}.$$

7. **Answer (C):** When Crystal travels one mile northeast she travels $\frac{\sqrt{2}}{2}$ miles north and $\frac{\sqrt{2}}{2}$ miles east. Similarly, when she travels southeast for one mile she travels $\frac{\sqrt{2}}{2}$ miles south and $\frac{\sqrt{2}}{2}$ miles east. Just before the last portion of her run she has traveled a net of $1 + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = 1$ miles north, and $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$ miles east. By the Pythagorean Theorem, the last portion of her run is

$$\sqrt{1^2 + (\sqrt{2})^2} = \sqrt{1 + 2} = \sqrt{3} \quad \text{miles.}$$

8. **Answer (D):** Tony worked for $2 \cdot 50 = 100$ hours. His average earnings per hour during this period is $\frac{\$630}{100} = \$6.30$. Hence his average age during this period was $\frac{\$6.30}{\$0.50} = 12.6$, and so at the end of the six month period he was 13.

9. **Answer (E):** Let $x + 32$ be written in the form $CDDC$. Because x has three digits, $1000 < x + 32 < 1032$, and so $C = 1$ and $D = 0$. Hence $x = 1001 - 32 = 969$, and the sum of the digits of x is $9 + 6 + 9 = 24$.

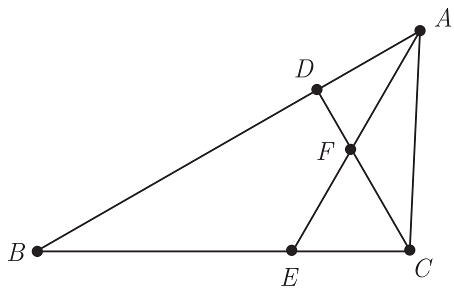
10. **Answer (E):** A non-leap year has 365 days, and $365 = 52 \cdot 7 + 1$, so there are 52 weeks and 1 day in a non-leap year. Because May 27 was after leap day in 2008, Marvin's birthday fell on Wednesday in 2009, and will fall on Thursday in 2010 and Friday in 2011. His birthday will be on Sunday in the leap year 2012, Monday in 2013, Tuesday in 2014, Wednesday in 2015, Friday in 2016, and Saturday in 2017.

11. **Answer (D):** The solution of the inequality is
$$\frac{a-3}{2} \le x \le \frac{b-3}{2}.$$
If $\frac{b-3}{2} - \frac{a-3}{2} = 10$, then $b - a = 20$.

12. **Answer (C):** The volume scale for Logan's model is $0.1 : 100{,}000 = 1 : 1{,}000{,}000$. Therefore the linear scale is $1 : \sqrt[3]{1{,}000{,}000}$, which is $1 : 100$. Logan's water tower should stand $\frac{40}{100} = 0.4$ meters tall.

13. **Answer (A):** Angelina drove $80t$ km before she stopped. After her stop, she drove $(3 - \frac{1}{3} - t)$ hours at an average rate of 100 kph, so she covered $100(\frac{8}{3} - t)$ km in that time. Therefore $80t + 100(\frac{8}{3} - t) = 250$. Note that $t = \frac{5}{6}$.s

14. **Answer (C):** Let $\alpha = \angle BAE = \angle ACD = \angle ACF$. Because $\triangle CFE$ is equilateral, it follows that $\angle CFA = 120^\circ$ and then
$$\angle FAC = 180^\circ - 120^\circ - \angle ACF = 60^\circ - \alpha.$$
Therefore
$$\angle BAC = \angle BAE + \angle FAC = \alpha + (60^\circ - \alpha) = 60^\circ.$$
Because $AB = 2 \cdot AC$, it follows that $\triangle BAC$ is a $30-60-90^\circ$ triangle, and thus $\angle ACB = 90^\circ$.



15. **Answer (D):** LeRoy and Chris cannot both be frogs, because their statements would be true and frogs lie. Also LeRoy and Chris cannot both be toads, because then their statements would be false, and toads tell the truth. Hence between LeRoy and Chris, exactly one must be a toad.

    If Brian is a toad, then Mike must be a frog, but this is a contradiction as Mike's statement would then be true. Hence Brian is a frog, so Brian's statement must be false, and Mike must be a frog. Altogether there are 3 frogs: Brian, Mike, and either LeRoy or Chris.

16. **Answer (B):** By the Angle Bisector Theorem, $8 \cdot BA = 3 \cdot BC$. Thus BA must be a multiple of 3. If $BA = 3$, the triangle is degenerate. If $BA = 6$, then $BC = 16$, and the perimeter is $6 + 16 + 11 = 33$.

17. **Answer (A):** The volume of the solid cube is 27 in^3. The first hole to be cut removes $2 \times 2 \times 3 = 12$ in^3 from the volume. The other holes remove $2 \times 2 \times 0.5 = 2$ in^3 from each of the four remaining faces. The volume of the remaining solid is $27 - 12 - 4(2) = 7$ in^3.

18. **Answer (B):** The probability that Bernardo picks a 9 is $\frac{3}{9} = \frac{1}{3}$. In this case, his three-digit number will begin with a 9 and will be larger than Silvia's three-digit number.

    If Bernardo does not pick a 9, then Bernardo and Silvia will form the same number with probability

    $$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

    If they do not form the same number then Bernardo's number will be larger $\frac{1}{2}$ of the time.

Hence the probability is

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}(1 - \frac{1}{56}) = \frac{111}{168} = \frac{37}{56}.$$

19. **Answer (E):** Triangles ABC, CDE and EFA are congruent, so $\triangle ACE$ is equilateral. Let X be the intersection of the lines AB and EF and define Y and Z similarly as shown in the figure. Because $ABCDEF$ is equiangular, it follows that $\angle XAF = \angle AFX = 60°$. Thus $\triangle XAF$ is equilateral. Let H be the midpoint of $\overline{XF}$. By the Pythagorean Theorem,

$$AE^2 = AH^2 + HE^2 = (\frac{\sqrt{3}}{2}r)^2 + (\frac{r}{2} + 1)^2 = r^2 + r + 1$$

Thus, the area of $\triangle ACE$ is

$$\frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1).$$

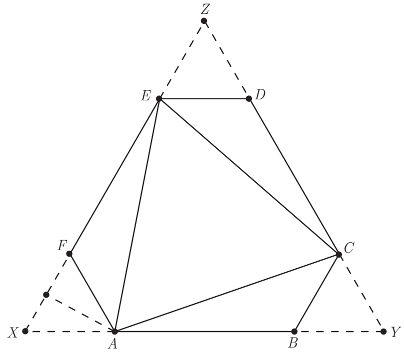
The area of hexagon $ABCDEF$ is equal to

$$[XYZ] - [XAF] - [YCB] - [ZED] = \frac{\sqrt{3}}{4}\left((2r+1)^2 - 3r^2\right) = \frac{\sqrt{3}}{4}(r^2 + 4r + 1)$$

Because $[ACE] = \frac{7}{10}[ABCDEF]$, it follows that

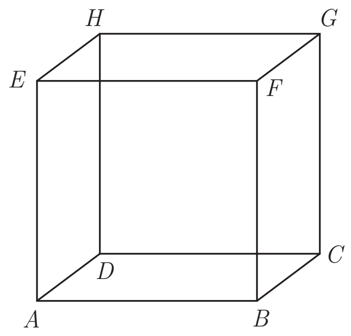
$$r^2 + r + 1 = \frac{7}{10}(r^2 + 4r + 1)$$

from which $r^2 - 6r + 1 = 0$ and $r = 3 \pm 2\sqrt{2}$. The sum of all possible values of r is 6.



20. **Answer (D):** Each of the 8 lines segments on the fly's path is an edge, a face diagonal, or an interior diagonal of the cube. These three type of line segments have lengths 1, $\sqrt{2}$, and $\sqrt{3}$, respectively. Because each vertex of the cube is visited only once, the two line segments that meet at a vertex have a combined length of at most $\sqrt{2}+\sqrt{3}$. Therefore the sum of the lengths of the 8 segments is at most $4\sqrt{2}+4\sqrt{3}$. This maximum is achieved by the path

$$A \to G \to B \to H \to C \to E \to D \to F \to A.$$



21. **Answer (A):** Let the polynomial be $(x-r)(x-s)(x-t)$ with $0 < r \le s \le t$. Then $rst = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$, and $r+s+t = a$. If $t = 67$, then $rs = 30$, and $r+s$ is minimized when $r = 5$ and $s = 6$. In that case $a = 67+5+6 = 78$. If $t \ne 67$, then $a > t \ge 2 \cdot 67 = 134$, so the minimum value of a is 78.

22. **Answer (A):** Three chords create a triangle if and only if they intersect pairwise inside the circle. Two chords intersect inside the circle if and only if their endpoints alternate in order around the circle. Therefore, if points A, B, C, D, E, and F are in order around the circle, then only the chords $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ all intersect pairwise inside the circle. Thus every set of 6 points determines a unique triangle, and there are $\binom{8}{6} = 28$ such triangles.

23. **Answer (A):** If Isabella reaches the k^{th} box, she will draw a white marble from it with probability $\frac{k}{k+1}$. For $n \ge 2$, the probability that she will draw white marbles from each of the first $n-1$ boxes is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n},$$

so the probability that she will draw her first red marble from the n^{th} box is $P(n) = \frac{1}{n(n+1)}$. The condition $P(n) < 1/2010$ is equivalent to $n^2+n-2010 > 0$, from which $n > \frac{1}{2}(-1+\sqrt{8041})$ and $(2n+1)^2 > 8041$. The smallest positive odd integer whose square exceeds 8041 is 91, and the corresponding value of n is 45.

24. **Answer (A):** There are 18 factors of 90! that are multiples of 5, 3 factors that are multiples of 25, and no factors that are multiples of higher powers of 5. Also, there are more than 45 factors of 2 in 90!. Thus $90! = 10^{21}N$ where N is an integer not divisible by 10, and if $N \equiv n \pmod{100}$ with $0 < n \le 99$, then n is a multiple of 4.

    Let $90! = AB$ where A consists of the factors that are relatively prime to 5 and B consists of the factors that are divisible by 5. Note that $\prod_{j=1}^{4}(5k+j) \equiv 5k(1+2+3+4)+1\cdot 2\cdot 3\cdot 4 \equiv 24 \pmod{25}$, thus

$$\begin{aligned} A &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot\cdots\cdot(86\cdot 87\cdot 88\cdot 89) \\ &\equiv 24^{18} \equiv (-1)^{18} \equiv 1 \pmod{25}. \end{aligned}$$

    Similarly,

$$B = (5\cdot 10\cdot 15\cdot 20)\cdot(30\cdot 35\cdot 40\cdot 45)\cdot(55\cdot 60\cdot 65\cdot 70)\cdot(80\cdot 85\cdot 90)\cdot(25\cdot 50\cdot 75),$$

    thus

$$\begin{aligned} \frac{B}{5^{21}} &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot(11\cdot 12\cdot 13\cdot 14)\cdot(16\cdot 17\cdot 18)\cdot(1\cdot 2\cdot 3) \\ &\equiv 24^3\cdot(-9)\cdot(-8)\cdot(-7)\cdot 6 \equiv (-1)^3\cdot 1 \equiv -1 \pmod{25}. \end{aligned}$$

    Finally, $2^{21} = 2\cdot(2^{10})^2 = 2\cdot(1024)^2 \equiv 2\cdot(-1)^2 \equiv 2 \pmod{25}$, so $13\cdot 2^{21} \equiv 13\cdot 2 \equiv 1 \pmod{25}$. Therefore

$$\begin{aligned} N &\equiv (13\cdot 2^{21})N = 13\cdot\frac{90!}{5^{21}} = 13\cdot A\cdot\frac{B}{5^{21}} \equiv 13\cdot 1\cdot(-1) \pmod{25} \\ &\equiv -13 \equiv 12 \pmod{25}. \end{aligned}$$

    Thus n is equal to 12, 37, 62, or 87, and because n is a multiple of 4, it follows that $n = 12$.

25. **Answer (B):** Let the sequence be $(a_1, a_2, \ldots, a_8)$. For $j > 1$, $a_{j-1} = a_j + m^2$ for some m such that $a_j < (m+1)^2 - m^2 = 2m+1$. To minimize the value of a_1, construct the sequence in reverse order and choose the smallest possible value of m for each j, $2 \le j \le 8$. The terms in reverse order are $a_8 = 0$, $a_7 = 1$, $a_6 = 1 + 1^2 = 2$, $a_5 = 2 + 1^2 = 3$, $a_4 = 3 + 2^2 = 7$, $a_3 = 7 + 4^2 = 23$, $a_2 = 23 + 12^2 = 167$, and $N = a_1 = 167 + 84^2 = 7223$, which has the unit digit 3.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Blasberg, Steven Davis, Sundeep Desai, Steven Dunbar, Sister Josannae Furey, David Grabiner, Michelle Ghrist, Jerrold Grossman, Brian Hartwig, Dan Kennedy, Joe Kennedy, Mike Korn, Leon La Spina, Glen Marr, Raymond Scacalossi, David Wells, LeRoy Wenstrom, Woody Wenstrom, and Ron Yannone.