MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

11th Annual

AMC 10 B

American Mathematics Contest 10B

Solutions Pamphlet

Wednesday, February 24, 2010

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (C):** Simplifying gives

$$\begin{aligned}100(100-3)-(100\cdot 100-3) &= 100\cdot 100-100\cdot 3-100\cdot 100+3\\ &= -300+3\\ &= -297.\end{aligned}$$

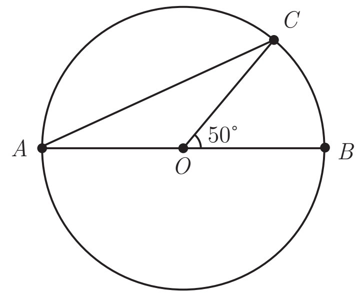
2. **Answer (C):** Makayla spent $45+2\cdot 45=135$ minutes, or $\frac{135}{60}=\frac{9}{4}$ hours in meetings. Hence she spent $100\cdot\frac{9/4}{9}=25$ percent of her time in meetings.

3. **Answer (C):** If a set of 4 socks does not contain a pair, there must be one of each color. The fifth sock must match one of the others and guarantee a matching pair.

4. **Answer (C):** Note that $\heartsuit(1)=\dfrac{1+1^2}{2}=1$, $\heartsuit(2)=\dfrac{2+2^2}{2}=3$, and $\heartsuit(3)=\dfrac{3+3^2}{2}=6$. Thus $\heartsuit(1)+\heartsuit(2)+\heartsuit(3)=1+3+6=10$.

5. **Answer (B):** A month with 31 days has 3 successive days of the week appearing five times and 4 successive days of the week appearing four times. If Monday and Wednesday appear five times then Monday must be the first day of the month. If Monday and Wednesday appear only four times then either Thursday or Friday must be the first day of the month. Hence there are 3 days of the week that could be the first day of the month.

6. **Answer (B):** Note that $\angle AOC=180^\circ-50^\circ=130^\circ$. Because $\triangle AOC$ is isosceles, $\angle CAB=\frac{1}{2}(180^\circ-130^\circ)=25^\circ$.



OR

By the Inscribed Angle Theorem, $\angle CAB = \frac{1}{2}(\angle COB) = \frac{1}{2}(50^\circ) = 25^\circ$.

7. **Answer (D):** Let the triangle be ABC with $AB = 12$, and let D be the foot of the altitude from C. Then $\triangle ACD$ is a right triangle with hypotenuse $AC = 10$ and one leg $AD = \frac{1}{2}AB = 6$. By the Pythagorean Theorem $CD = \sqrt{10^2 - 6^2} = 8$, and the area of $\triangle ABC$ is $\frac{1}{2}(AB)(CD) = \frac{1}{2}(12)(8) = 48$. The rectangle has length $\frac{48}{4} = 12$ and perimeter $2(12 + 4) = 32$.

8. **Answer (E):** The cost of an individual ticket must divide 48 and 64. The common factors of 48 and 64 are $1, 2, 4, 8$, and 16. Each of these may be the cost of one ticket, so there are 5 possible values for x.

9. **Answer (D):** The correct answer was $1-(2-(3-(4+e))) = 1-2+3-4-e = -2-e$. Larry's answer was $1-2-3-4+e = -8+e$. Therefore $-2-e = -8+e$, so $e = 3$.

10. **Answer (C):** Let t be the number of minutes Shelby spent driving in the rain. Then she traveled $20\frac{t}{60}$ miles in the rain, and $30\frac{40-t}{60}$ miles in the sun. Solving $20\frac{t}{60} + 30\frac{40-t}{60} = 16$ results in $t = 24$ minutes.

11. **Answer (A):** Let p dollars be the purchase price of the stem. The savings provided by Coupon A, B, and C respectively are $0.15p$, 30, and $0.25(p - 100)$. Coupon A saves at least as much as Coupon B if $0.15p \geq 30$, so $p \geq 200$. Coupon A saves at least as much as Coupon C if $0.15p \geq 0.25(p - 100)$, so $p \leq 250$. Therefore $x = 200$, $y = 250$, and $y - x = 50$.

12. **Answer (D):** Assume there are 100 students in Mr. Wells' class. Then at least $70 - 50 = 20$ students answered "No" at the beginning of the school year and "Yes" at the end, so $x \geq 20$. Because only 30 students answered "No" at the end of the school year, at least $50-30 = 20$ students who answered "Yes"' at the beginning of the year gave the same answer at the end, so $x \leq 80$. The difference between the maximum and minimum possible values of x is $80 - 20 = 60$. The minimum $x = 20$ is achieved if exactly 20 students answered "No" at the beginning and "Yes" at the end of the school year. The maximum $x = 80$ is achieved if exactly 20 students answered "Yes at the beginning and the end.

13. **Answer (C):** If $60 - 2x > 0$, then $|2x - |60 - 2x|| = |4x - 60|$. Solving $x = 4x - 60$, and $x = -(4x - 60)$ results in $x = 20$, and $x = 12$, respectively, both of which satisfy the original equation.
If $60 - 2x < 0$, then $|2x - |60 - 2x|| = |2x + 60 - 2x| = 60$. Note that $x = 60$ satisfies the original equation. The sum of the solutions is $12 + 20 + 60 = 92$.

14. **Answer (B):** The average of the numbers is

$$\frac{1 + 2 + \cdots + 99 + x}{100} = \frac{\frac{99 \cdot 100}{2} + x}{100} = \frac{99 \cdot 50 + x}{100} = 100x.$$

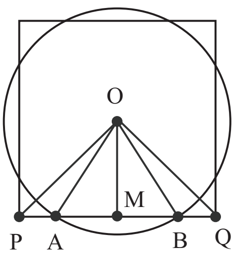
This equation is equivalent to $9999x = (99 \cdot 101)x = 99 \cdot 50$, so $x = \frac{50}{101}$.

15. **Answer (C):** If Jesse answered R questions correctly and W questions incorrectly, then $R + W \leq 50$, and Jesse's score is $99 = 4R - W \geq 4R - (50 - R) = 5R - 50$. Thus $5R \leq 149$, and because R is an integer, $R \leq 29$, Jesse could achieve a score of 99 by answering 29 questions correctly and 17 incorrectly, leaving 4 answers blank.

16. **Answer (B):** Let O be the common center of the circle and the square. Let M be the midpoint of a side of the square and P and Q be the vertices of the square on the side containing M. Since

$$OM^2 = \left(\frac{1}{2}\right)^2 < \left(\frac{\sqrt{3}}{3}\right)^2 < \left(\frac{\sqrt{2}}{2}\right)^2 = OP^2 = OQ^2,$$

the midpoint of each side is inside the circle and the vertices of the square are outside the circle. Therefore the circle intersects the square in two points along each side.



Let A and B be the intersection points of the circle with $\overline{PQ}$. Then M is also the midpoint of $\overline{AB}$ and $\triangle OMA$ is a right triangle. By the Pythagorean Theorem $AM = \frac{1}{2\sqrt{3}}$, so $\triangle OMA$ is a $30-60-90°$ right triangle. Then $\angle AOB = 60°$, and the area of the sector corresponding to $\angle AOB$ is $\frac{1}{6} \cdot \pi \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\pi}{18}$. The area of $\triangle AOB$ is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{12}$. The area outside the square but inside the circle is $4 \cdot (\frac{\pi}{18} - \frac{\sqrt{3}}{12}) = \frac{2\pi}{9} - \frac{\sqrt{3}}{3}$.

17. **Answer (B):** If there are n schools in the city, then there are $3n$ contestants, so $3n \geq 64$, and $n \geq 22$. Because Andrea received the median score and each student received a different score, n is odd, so $n \geq 23$. Andrea's position is $\frac{3n+1}{2}$, and Andrea finished ahead of Beth, so $\frac{3n+1}{2} < 37$, and $3n < 73$. Because n is an odd integer, $n \leq 23$. Therefore $n = 23$.

18. **Answer (E):** Let $N = abc + ab + a = a(bc + b + 1)$. If a is divisible by 3, then N is divisible by 3. Note that 2010 is divisible by 3, so the probability that a is divisible by 3 is $\frac{1}{3}$.

    If a is not divisible by 3 then N is divisible by 3 if $bc + b + 1$ is divisible by 3. Define b_0 and b_1 so that $b = 3b_0 + b_1$ is an integer and b_1 is equal to 0, 1, or 2. Note that each possible value of b_1 is equally likely. Similarly define c_0 and c_1. Then

$$\begin{aligned} bc + b + 1 &= (3b_0 + b_1)(3c_0 + c_1) + 3b_0 + b_1 + 1 \\ &= 3(3b_0c_0 + c_0b_1 + c_1b_0 + b_0) + b_1c_1 + b_1 + 1. \end{aligned}$$

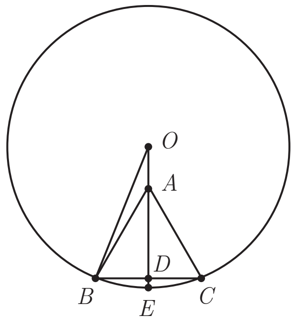
    Hence $bc + b + 1$ is divisible by 3 if and only if $b_1 = 1$ and $c_1 = 1$, or $b_1 = 2$ and $c_1 = 0$. The probability of this occurrence is $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

    Therefore the requested probability is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$.

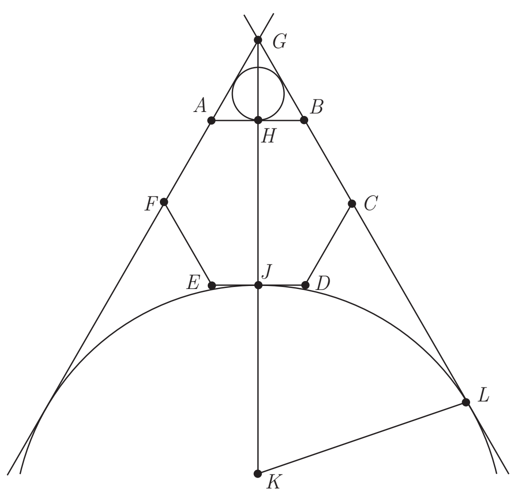
19. **Answer (B):** The radius of circle O is $\sqrt{156} > 4\sqrt{3} = OA$, so A is inside the circle. Let s be the side length of $\triangle ABC$, let D be the foot of the altitude from A, and let $\overline{OE}$ be the radius through A. This radius is perpendicular to $\overline{BC}$ and contains D, so $OD = \sqrt{OB^2 - BD^2} = \sqrt{156 - \frac{1}{4}s^2}$. If A is on $\overline{DE}$, then $\angle BAC > \angle BEC > 90^\circ$, an impossibility. Therefore A lies on $\overline{OD}$, and $OA = OD - AD$, that is,

$$4\sqrt{3} = \sqrt{156 - \frac{1}{4}s^2} - \frac{\sqrt{3}}{2}s.$$

    Rearranging terms and squaring both sides leads to the quadratic equation $s^2 + 12s - 108 = 0$, and the positive solution is $s = 6$.



20. **Answer (D):** It may be assumed that hexagon $ABCDEF$ has side length 1. Let lines BC and FA intersect at G, let H and J be the midpoints of $\overline{AB}$ and $\overline{DE}$, respectively, let K be the center of the second circle, and let that circle be tangent to line BC at L. Equilateral $\triangle ABG$ has side length 1, so the first circle, which is the inscribed circle of $\triangle ABG$, has radius $\frac{\sqrt{3}}{6}$. Let r be the radius of the second circle. Then $\triangle GLK$ is a $30-60-90^\circ$ right triangle with $LK = r$ and $2r = GK = GH + HJ + JK = \frac{\sqrt{3}}{2} + \sqrt{3} + r$. Therefore $r = \frac{3\sqrt{3}}{2} = 9(\frac{\sqrt{3}}{6})$. The ratio of the radii of the two circles is 9, and the ratio of their areas is $9^2 = 81$.



21. **Answer (E):** Each four-digit palindrome has digit representation $abba$ with $1 \le a \le 9$ and $0 \le b \le 9$. The value of the palindrome is $1001a + 110b$. Because 1001 is divisible by 7 and 110 is not, the palindrome is divisible by 7 if and only if $b = 0$ or $b = 7$. Thus the requested probability is $\frac{2}{10} = \frac{1}{5}$.

22. **Answer (C):** If there were no restrictions on the number of candies per bag, then each piece of candy could be distributed in 3 ways. In this case there would be 3^7 ways to distribute the candy. However, this counts the cases where the red bag or blue bag is empty.

    If the red bag remained empty then the candy could be distributed in 2^7 ways. The same is true for the blue bag. Both totals include the case in which all the candy is put into the white bag. Hence there are $2^7 + 2^7 - 1$ ways to distribute the candy such that either the red or blue bag is empty.

    The number of ways to distribute the candy, subject to the given conditions, is $3^7 - (2^7 + 2^7 - 1) = 1932$.

23. **Answer (D):** Let a_{ij} denote the entry in row i and column j. he given conditions imply that $a_{11} = 1$, $a_{33} = 9$, and $a_{22} = 4, 5$, or 6. If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$, and the sets $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$ are complementary subsets of $\{5, 6, 7, 8\}$. There are $\binom{4}{2} = 6$ ways to choose $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$, and only one way to order the entries. There are 2 ways to order $\{a_{12}, a_{21}\}$, so 12 arrays with $a_{22} = 4$ meet the given conditions. Similarly, the conditions are met by 12 arrays with $a_{22} = 6$. If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the conditions $a_{12} < 5$, $a_{21} < 5$, $a_{32} > 5$, and $a_{23} > 5$. Thus $\{a_{12}, a_{13}, a_{23}\} \neq \{2, 3, 4\}$ or $\{6, 7, 8\}$, so its elements can be chosen in $\binom{6}{3} - 2 = 18$ ways. Both the remaining entries and the ordering of all entries are then determined, so 18 arrays with $a_{22} = 5$ meet the given conditions.

    Altogether, the conditions are met by $12 + 12 + 18 = 42$ arrays.

24. **Answer (E):** The Raiders' score was $a(1 + r + r^2 + r^3)$, where a is a positive integer and $r > 1$. Because ar is also an integer, $r = m/n$ for relatively prime positive integers m and n with $m > n$. Moreover $ar^3 = a \cdot \frac{m^3}{n^3}$ is an integer, so n^3 divides a. Let $a = n^3 A$. Then the Raiders' score was $R = A(n^3 + mn^2 + m^2 n + m^3)$, and the Wildcats' score was $R-1 = a+(a+d)+(a+2d)+(a+3d) = 4a+6d$ for some positive integer d. Because $A \geq 1$, the condition $R \leq 100$ implies that $n \leq 2$ and $m \leq 4$. The only possibilities are $(m, n) = (4, 1), (3, 2), (3, 1), (4, 1)$, or $(2, 1)$. The corresponding values of R are, respectively, $85A$, $65A$, $40A$, and $15A$. In the first two cases $A = 1$, and the corresponding values of $R - 1$ are, respectively, $64 = 32+6d$ and $84 = 4+6d$. In neither case is d an integer. In the third case $40A = 40a = 4a + 6d + 1$ which is impossible in integers. In the last case $15a = 4a + 6d + 1$, from which $11a = 6d + 1$. The only solution in positive integers for which $4a+6d \leq 100$ is $(a, d) = (5, 9)$. Thus $R = 5+10+20+40 = 75$, $R - 1 = 5 + 14 + 23 + 32 = 74$, and the number of points scored in the first half was $5 + 10 + 5 + 14 = 34$.

25. **Answer (B):** Because $1, 3, 5$, and 7 are roots of the polynomial $P(x) - a$, it follows that

$$P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. The previous identity must hold for $x = 2, 4, 6$, and 8, thus

$$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$$

Therefore $315 = \text{lcm}(15, 9, 105)$ divides a, that is a is an integer multiple of 315. Let $a = 315A$. Because $Q(2) = Q(6) = 42A$, it follows that $Q(x) - 42A = (x-2)(x-6)R(x)$ where $R(x)$ is a polynomial with integer coefficients. Because $Q(4) = -70A$ and $Q(8) = -6A$ it follows that $-112A = -4R(4)$ and $-48A = 12R(8)$, that is $R(4) = 28A$ and $R(8) = -4A$. Thus $R(x) = 28A + (x-4)(-6A + (x-8)T(x))$ where $T(x)$ is a polynomial with integer coefficients. Moreover, for any polynomial $T(x)$ and any integer A, the polynomial $P(x)$ constructed this way satisfies the required conditions. The required minimum is obtained when $A = 1$ and so $a = 315$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Blasberg, Steven Davis, Steven Dunbar, Douglas Faires, Michelle Ghrist, Jerrold Grossman, Joe Kennedy, Leon La Spina, Raymond Scacalossi, William Wardlaw, David Wells, LeRoy Wenstrom, Woody Wenstrom and Ron Yannone.

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