MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

12th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 8, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (D):** The text messages cost $\$0.05 \cdot 100 = \$5.00$, and the 30 minutes of excess chatting cost $\$0.10 \cdot 30 = \$3.00$. Therefore the total bill came to $\$5 + \$3 + \$20 = \$28$.

2. **Answer (E):** Because $14 \cdot 35 = 490 < 500$ and $15 \cdot 35 = 525 \geq 500$, the minimum number of bottles that she needs to buy is 15.

3. **Answer (D):** First note that $\{1\ 1\ 0\} = \frac{2}{3}$ and $[0\ 1] = \frac{1}{2}$. Therefore

$$\{\{1\ 1\ 0\}\ [0\ 1]\ 0\} = \{\frac{2}{3}\ \frac{1}{2}\ 0\} = \frac{\frac{2}{3} + \frac{1}{2} + 0}{3} = \frac{7}{18}.$$

4. **Answer (A):** Every term in X except 10 appears in Y. Every term in Y except 102 appears in X. Therefore $Y - X = 102 - 10 = 92$.

OR

The sum X has 46 terms because it includes all 50 even positive integers less than or equal to 100 except for 2, 4, 6, and 8. The sum Y has the same number of terms, and every term in Y exceeds the corresponding term in X by 2. Therefore $Y - X = 46 \cdot 2 = 92$.

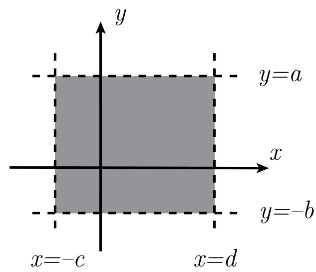
5. **Answer (C):** Let N equal the number of fifth graders. Then there are $2N$ fourth graders and $4N$ third graders. The total number of minutes run per day by the students is $4N \cdot 12 + 2N \cdot 15 + N \cdot 10 = 88N$. There are a total of $4N + 2N + N = 7N$ students, so the average number of minutes run by the students per day is $\frac{88N}{7N} = \frac{88}{7}$.

6. **Answer (C):** The union must contain all of the elements of A, so it has at least 20 elements. It is possible that B is a subset of A, in which case there are no additional elements.

7. **Answer (B):** Because $|-3x|+5$ is strictly positive, the equation $|-3x|+5 = 0$ has no solution. The solutions of equations **(A)**, **(C)**, **(D)**, and **(E)** are -7, -4, 64, and $\pm\frac{4}{3}$, respectively.

8. **Answer (C):** Because 75% of the birds were not swans and 30% of the birds were geese, it follows that $\frac{30}{75} \cdot 100\% = 40\%$ of the birds that were not swans were geese.

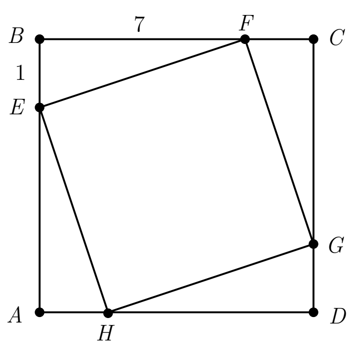
9. **Answer (A):** Because a, b, c, and d are positive numbers, $a > -b$ and $d > -c$. Therefore the height of the rectangle is $a + b$ and the width is $c + d$. The area of the region is $(a + b)(c + d) = ac + ad + bc + bd$.



10. **Answer (B):** Let C be the cost of a pencil in cents, N be the number of pencils each student bought, and S be the number of students who bought pencils. Then $C \cdot N \cdot S = 1771 = 7 \cdot 11 \cdot 23$, and $C > N > 1$. Because a majority of the students bought pencils, $30 \ge S > \frac{30}{2} = 15$. Therefore $S = 23, N = 7$, and $C = 11$.

11. **Answer (B):** Without loss of generality, assume that F lies on $\overline{BC}$ and that $EB = 1$. Then $AE = 7$ and $AB = 8$. Because $EFGH$ is a square, $BF = AE = 7$, so the hypotenuse $\overline{EF}$ of $\triangle EBF$ has length $\sqrt{1^2 + 7^2} = \sqrt{50}$. The ratio of the area of $EFGH$ to that of $ABCD$ is therefore

$$\frac{EF^2}{AB^2} = \frac{50}{64} = \frac{25}{32}.$$



12. **Answer (A):** Let x, y, and z be the number of successful three-point shots, two-point shots, and free throws, respectively. Then the given conditions imply

$$\begin{aligned} 3x + 2y + z &= 61, \\ 2y &= 3x, \text{ and} \\ y + 1 &= z. \end{aligned}$$

Solving results in $x = 8$, $y = 12$, and $z = 13$. Hence the team made 13 free throws.

13. **Answer (A):** Because the numbers are even, they must end in either 2 or 8. If the last digit is 2, the first digit must be 5 and thus there are four choices remaining for the middle digit. If the last digit is 8, then there are two choices for the first digit, either 2 or 5, and for each choice there are four possibilities for the middle digit. The total number of choices is then $4 + 2 \cdot 4 = 12$.

14. **Answer (B):** Let d be the sum of the numbers rolled. The conditions are satisfied if and only if $\pi\left(\frac{d}{2}\right)^2 < \pi d$, that is, $d < 4$. Of the 36 equally likely outcomes for the roll of the two dice, one has a sum of 2 and two have sums of 3. Thus the desired probability is $\frac{1+2}{36} = \frac{1}{12}$.

15. **Answer (C):** Let x be the number of miles driven exclusively on gasoline. Then the total number of miles traveled is $x + 40$, and the amount of gas used is $0.02x$ gallons. Therefore the average number of miles per gallon is

$$\frac{x + 40}{0.02x} = 55\,.$$

Solving results in $x = 400$, so the total number of miles traveled is 440.

16. **Answer (B):** Let $k = \sqrt{9 - 6\sqrt{2}} + \sqrt{9 + 6\sqrt{2}}$. Squaring both sides and simplifying results in

$$\begin{aligned} k^2 &= 9 - 6\sqrt{2} + 2\sqrt{(9 - 6\sqrt{2})(9 + 6\sqrt{2})} + 9 + 6\sqrt{2} \\ &= 18 + 2\sqrt{81 - 72} \\ &= 18 + 2\sqrt{9} \\ &= 24 \end{aligned}$$

Because $k > 0$, $k = 2\sqrt{6}$.

17. **Answer (C):** Note that for any four consecutive terms, the first and last terms must be equal. For example, consider B, C, D, and E; because

$$B + C + D = 30 = C + D + E\,,$$

we must have $B = E$. Hence $A = D = G$, and $C = F = 5$. The required sum $A + H = G + (30 - G - F) = 30 - 5 = 25$.

OR

Note that

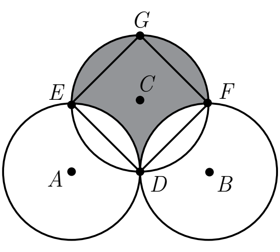
$$\begin{aligned} A + C + H &= (A + B + C) - (B + C + D) + (C + D + E) \\ &\quad - (E + F + G) + (F + G + H) \\ &= 3 \cdot 30 - 2 \cdot 30 = 30. \end{aligned}$$

Hence $A + H = 30 - C = 25$.

18. **Answer (C):** Let D be the midpoint of $\overline{AB}$, and let circle C intersect circles A and B at E and F, respectively, distinct from D. The shaded portion of unit square $ADCE$ has area $1 - \frac{\pi}{4}$, as does the shaded portion of unit square $BDCF$. The portion of the shaded region which is outside these squares is a semicircle of radius 1 and has area $\frac{\pi}{2}$. The total shaded area is $2\left(1 - \frac{\pi}{4}\right) + \frac{\pi}{2} = 2$.

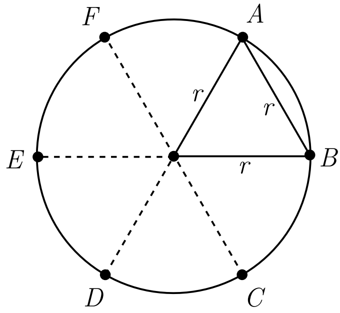
OR

Let D, E, and F be defined as in the first solution, and let G be diametrically opposite D on circle C. The shaded area is equal to the area of square $DFGE$, which has diagonal length 2. Its side length is $\sqrt{2}$, and its area is $\left(\sqrt{2}\right)^2 = 2$.



19. **Answer (E):** Let p^2, $q^2 + 9$, and $r^2 = p^2 + 300$ be the populations of the town in 1991, 2001, and 2011, respectively. Then $q^2 + 9 = p^2 + 150$, so $q^2 - p^2 = 141$. Therefore $(q - p)(q + p) = 141$, and so either $q - p = 3$ and $q + p = 47$, or $q - p = 1$ and $q + p = 141$. These give $p = 22$ or $p = 70$. Note that if $p = 70$, then $70^2 + 300 = 5200 = 52 \cdot 10^2$, which is not a perfect square. Thus $p = 22$, $p^2 = 484$, $p^2 + 150 = 634 = 25^2 + 9$, and $p^2 + 300 = 784 = 28^2$. The percent growth from 1991 to 2011 was $\frac{784-484}{484} \approx 62\%$.

20. **Answer (D):** Let point A be the first point chosen, and let point B be the opposite endpoint of the corresponding chord. Drawing a radius to each endpoint of this chord of length r results in an equilateral triangle. Hence a chord of length r subtends an arc $\frac{1}{6}$ the circumference of the circle. Let diameter $\overline{FC}$ be parallel to $\overline{AB}$, and divide the circle into six equal portions as shown. The second point chosen will result in a chord that intersects $\overline{AB}$ if and only if the point is chosen from minor $\widehat{FB}$. Hence the probability is $\frac{1}{3}$.



21. **Answer (D):** The weights of the two pairs of coins are equal if each pair contains the same number of counterfeit coins. Therefore either the first pair and the second pair both contain only genuine coins, or the first pair and the second pair both contain one counterfeit coin. The number of ways to choose the coins in the first case is $\binom{8}{2} \cdot \binom{6}{2} = 420$. The number of ways to choose the coins in the second case is $8 \cdot 2 \cdot 7 \cdot 1 = 112$. Therefore the requested probability is $\frac{420}{112+420} = \frac{15}{19}$.

22. **Answer (C):** If five distinct colors are used, then there are $\binom{6}{5} = 6$ different color choices possible. They may be arranged in $5! = 120$ ways on the pentagon, resulting in $120 \cdot 6 = 720$ colorings.

If four distinct colors are used, then there is one duplicated color, so there are $\binom{6}{4}\binom{4}{1} = 60$ different color choices possible. The duplicated color must appear on neighboring vertices. There are 5 neighbor choices and $3! = 6$ ways to color the remaining three vertices, resulting in a total of $60 \cdot 5 \cdot 6 = 1800$ colorings.

If three distinct colors are used, then there must be two duplicated colors, so there are $\binom{6}{3}\binom{3}{2} = 60$ different color choices possible. The non-duplicated color may appear in 5 locations. As before, a duplicated color must appear on neighboring vertices, so there are 2 ways left to color the remaining vertices. In this case there are $60 \cdot 5 \cdot 2 = 600$ colorings possible.

There are no colorings with two or fewer colors. The total number of colorings is $720 + 1800 + 600 = 3120$.

23. **Answer (C):** After each person counts, the numbers left for the next person form an arithmetic progression. For example, Alice leaves all of the numbers $2, 5, 8, 11, 14, \ldots, 2 + 3 \cdot 332$ for Barbara. If a student leaves the progression $a, a + d, a + 2d, a + 3d, a + 4d, \ldots$, then the next student leaves the progression $a + d, (a + d) + 3d, (a + d) + 6d, \ldots$.

This implies that in the following table, each number in the third column is three times the previous entry in the third column, and each entry in the second column is the sum of the two entries in the row above:

Left for	First Term	Common Difference
Alice	1	1
Barbara	2	3
Candice	5	9
Debbie	14	27
Eliza	41	81
Fatima	122	243
George	365	729

George is left with the single term 365.

OR

The numbers skipped by Alice are the middle numbers in each consecutive group of 3, that is, $2, 5, 8$, and so on. The numbers skipped by Alice and Barbara are the middle numbers in each group of 9, that is, $5, 14, 23$, and so on. In general, the numbers skipped by all of the first n students are the middle numbers in each group of 3^n. Because $3^6 = 729$, the only number not exceeding 1000 that is skipped by the first six students is $\frac{729+1}{2} = 365$. That is the number that George says.

24. **Answer (D):** Let the tetrahedra be T_1 and T_2, and let R be their intersection. Let squares $ABCD$ and $EFGH$, respectively, be the top and bottom faces of the unit cube, with E directly under A and F directly under B. Without loss of generality, T_1 has vertices A, C, F, and H, and T_2 has vertices B, D, E, and G. One face of T_1 is $\triangle ACH$, which intersects edges of T_2 at the midpoints J, K, and L of $\overline{AC}$, $\overline{CH}$, and $\overline{HA}$, respectively. Let S be the tetrahedron with vertices J, K, L, and D. Then S is similar to T_2 and is contained in T_2, but not in R. The other three faces of T_1 each cut off from T_2 a tetrahedron congruent to S. Therefore the volume of R is equal to the volume of T_2 minus four times the volume of S.

A regular tetrahedron of edge length s has base area $\frac{\sqrt{3}}{4}s^2$ and altitude $\frac{\sqrt{6}}{3}s$, so its volume is $\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right)\left(\frac{\sqrt{6}}{3}s\right) = \frac{\sqrt{2}}{12}s^3$. Because the edges of tetrahedron T_2 are face diagonals of the cube, T_2 has edge length $\sqrt{2}$. Because J and K are centers of adjacent faces of the cube, tetrahedron S has edge length $\frac{\sqrt{2}}{2}$. Thus the volume of R is

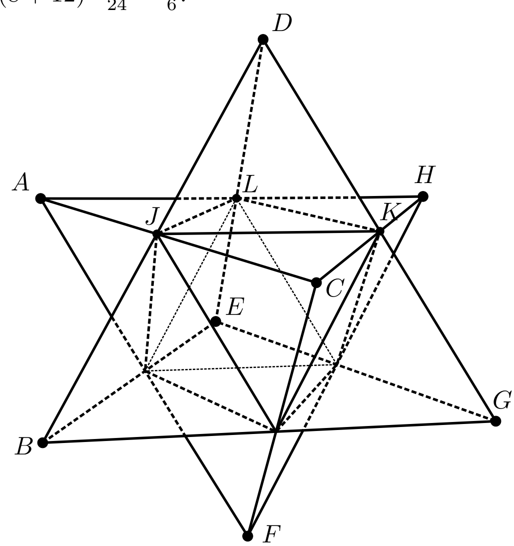
$$\frac{\sqrt{2}}{12}\left((\sqrt{2})^3 - 4\left(\frac{\sqrt{2}}{2}\right)^3\right) = \frac{1}{6}.$$

OR

Let T_1 and T_2 be labeled as in the previous solution. The cube is partitioned by T_1 and T_2 into 8 tetrahedra congruent to $DJKL$ (one for every vertex of the cube), 12 tetrahedra congruent to $AJLD$ (one for every edge of the cube), and the solid $T_1 \cap T_2$. Because the bases AJL and JLK are equilateral triangles with the same area, and the altitudes to vertex D of the tetrahedra $AJLD$ and $DJKL$ are the same, it follows that the volumes of $AJLD$ and $DJKL$ are equal. Moreover,

$$\text{Volume}(AJLD) = \frac{1}{3}\text{Area}(ALD) \cdot h_J,$$

where $h_J = \frac{1}{2}$ is the distance from J to the face ALD, and $\text{Area}(ALD) = \frac{1}{4}$. Therefore $\text{Volume}(AJLD) = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$, and thus the volume of $T_1 \cap T_2$ is equal to $1 - (8 + 12) \cdot \frac{1}{24} = \frac{1}{6}$.



25. **Answer (C):** Assume without loss of generality that R is bounded by the square with vertices $A = (0, 0)$, $B = (1, 0)$, $C = (1, 1)$, and $D = (0, 1)$, and let $X = (x, y)$ be n-ray partitional. Because the n rays partition R into triangles, they must include the rays from X to A, B, C, and D. Let the number of rays intersecting the interiors of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ be n_1, n_2, n_3, and n_4, respectively. Because $\triangle ABX \cup \triangle CDX$ has the same area as $\triangle BCX \cup \triangle DAX$,

it follows that $n_1+n_3 = n_2+n_4 = \frac{n}{2}-2$, so n is even. Furthermore, the n_1+1 triangles with one side on $\overline{AB}$ have equal area, so each has area $\frac{1}{2}\cdot\frac{1}{n_1+1}\cdot y$. Similarly, the triangles with sides on $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have areas $\frac{1}{2}\cdot\frac{1}{n_2+1}\cdot(1-x)$, $\frac{1}{2}\cdot\frac{1}{n_3+1}\cdot(1-y)$, and $\frac{1}{2}\cdot\frac{1}{n_4+1}\cdot x$, respectively. Setting these expressions equal to each other gives

$$x = \frac{n_4+1}{n_2+n_4+2} = \frac{2(n_4+1)}{n} \text{ and } y = \frac{n_1+1}{n_1+n_3+2} = \frac{2(n_1+1)}{n}.$$

Thus an n-ray partitional point must have the form $X = (\frac{2a}{n}, \frac{2b}{n})$ with $1 \le a < \frac{n}{2}$ and $1 \le b < \frac{n}{2}$. Conversely, if X has this form, R is partitioned into n triangles of equal area by the rays from X that partition $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ into b, $\frac{n}{2}-a$, $\frac{n}{2}-b$, and a congruent segments, respectively.

Assume X is 100-ray partitional. If X is also 60-ray partitional, then $X = (\frac{a}{50}, \frac{b}{50}) = (\frac{c}{30}, \frac{d}{30})$ for some integers $1 \le a, b \le 49$ and $1 \le c, d \le 29$. Thus $3a = 5c$ and $3b = 5d$; that is, both a and b are multiples of 5. Conversely, if a and b are multiples of 5, then

$$X = \left(\frac{a}{50}, \frac{b}{50}\right) = \left(\frac{\frac{3a}{5}}{30}, \frac{\frac{3b}{5}}{30}\right)$$

is 60-ray partitional. Because there are exactly 9 multiples of 5 between 1 and 49, the required number of points X is equal to $49^2 - 9^2 = 40 \cdot 58 = 2320$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Dunbar, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Torney, David Wells, LeRoy Wenstrom, and Ron Yannone.

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