MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

12th Annual

AMC 10 B

American Mathematics Contest 10 B

Wednesday, February 23, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

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1. **Answer (C):** The given expression is equal to

$$\frac{12}{9} - \frac{9}{12} = \frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} = \frac{7}{12}.$$

2. **Answer (E):** The sum of her first 5 test scores is 385, yielding an average of 77. To raise her average to 80, her 6^{th} test score must be the difference between $6 \cdot 80 = 480$ and 385, which is 95.

3. **Answer (A):** The smallest possible width for the rectangle is $2 - 0.5 = 1.5$ inches. Similarly the smallest possible length is 2.5 inches. Hence the minimum area is $(1.5)(2.5) = 3.75$ square inches.

4. **Answer (C):** Bernardo has paid $B - A$ dollars more than LeRoy. If LeRoy gives Bernardo half of that difference, $\frac{B-A}{2}$, then each will have paid the same amount.

5. **Answer (E):** Because $161 = 23 \cdot 7$, the only two digit factor of 161 is 23. The correct product must have been $32 \cdot 7 = 224$.

6. **Answer (A):** Let x be Casper's original number of candies. After the first day he was left with $x - (\frac{1}{3}x + 2) = \frac{2}{3}x - 2$ candies. On the second day he ate $\frac{1}{3}(\frac{2}{3}x - 2)$ candies, gave away 4 candies, and was left with 8 candies. Therefore

$$\frac{2}{3}x - 2 - \left(\frac{1}{3}\left(\frac{2}{3}x - 2\right) + 4\right) = 8.$$

Solving for x results in $x = 30$.

OR

Before giving 4 candies to his sister, Casper had 12. This was $\frac{2}{3}$ of the number he had after the first day, so he had 18 after the first day. Before giving 2 candies to his brother, he had 20, and this was $\frac{2}{3}$ of the number he had originally. Therefore he had 30 candies at the beginning.

7. **Answer (B):** The degree measures of two of the angles have a sum of $\frac{6}{5} \cdot 90 = 108$ and a positive difference of 30, so their measures are 69 and 39. The remaining angle has a degree measure of $180 - 108 = 72$, which is the largest angle.

8. **Answer (B):** Because the beach was not crowded on June 10, at least one of the conditions was not met. That is, the weather might have been cooler than $80\,^\circ F$ and sunny, at least $80\,^\circ F$ and cloudy, or cooler than $80\,^\circ F$ and cloudy. The first possibility shows that **(A)** and **(E)** are invalid, the second shows that **(C)** is invalid, and the third shows that **(D)** is invalid. Only conclusion **(B)** is consistent with all three possibilities.

9. **Answer (D):** The area of $\triangle ABC$ is $\frac{1}{2}\cdot 3\cdot 4 = 6$, so the area of $\triangle EBD$ is $\frac{1}{3}\cdot 6 = 2$. Note that $\triangle ABC$ and $\triangle EBD$ are right triangles with an angle in common, so they are similar. Therefore BD and DE are in the ratio 4 to 3. Let $BD = x$ and $DE = \frac{3}{4}x$. Then the area of $\triangle EBD$ can be expressed as $\frac{1}{2}\cdot x\cdot \frac{3}{4}x = \frac{3}{8}x^2$. Because $\triangle EBD$ has area 2, solving yields $BD = \frac{4\sqrt{3}}{3}$.

OR

Because $\triangle EBD$ and $\triangle ABC$ are similar triangles, their areas are in the ratio of the squares of their corresponding linear parts. Therefore $\left(\frac{BD}{4}\right)^2 = \frac{1}{3}$ and $BD = \frac{4\sqrt{3}}{3}$.

10. **Answer (B):** The sum of the smallest ten elements is

$$1 + 10 + 100 + \cdots + 1{,}000{,}000{,}000 = 1{,}111{,}111{,}111.$$

Hence the desired ratio is

$$\frac{10{,}000{,}000{,}000}{1{,}111{,}111{,}111} = \frac{9{,}999{,}999{,}999 + 1}{1{,}111{,}111{,}111} = 9 + \frac{1}{1{,}111{,}111{,}111} \approx 9.$$

OR

The sum of a finite geometric series of the form $a(1 + r + r^2 + \cdots + r^n)$ is $\frac{a}{1-r}(1 - r^{n+1})$. The desired denominator $1 + 10 + 10^2 + \cdots + 10^9$ is a finite geometric series with $a = 1, r = 10$, and $n = 9$. Therefore the ratio is

$$\frac{10^{10}}{1 + 10 + 10^2 + \cdots + 10^9} = \frac{10^{10}}{\frac{1}{1-10}(1 - 10^{10})} = \frac{10^{10}}{10^{10} - 1}\cdot 9 \approx \frac{10^{10}}{10^{10}}\cdot 9 = 9.$$

11. **Answer (D):** If no more than 4 people have birthdays in any month, then at most 48 people would be accounted for. Therefore the statement is true for $n = 5$. The statement is false for $n \geq 6$ if, for example, 5 people have birthdays in each of the first 4 months of the year, and 4 people have birthdays in each of the last 8 months, for a total of $5\cdot 4 + 4\cdot 8 = 52$ people.

OR

The average number of birthdays per month is $\frac{52}{12}$, which is strictly between 4 and 5. Therefore at least one month must contain at least 5 birthdays, and, as above, it is possible to distribute the birthdays so that all months contain 4 or 5 birthdays.

12. **Answer (A):** The only parts of the track that are longer walking on the outside edge rather than the inside edge are the two semicircular portions. If the radius of the inner semicircle is r, then the difference in the lengths of the two paths is $2\pi(r+6) - 2\pi r = 12\pi$. Let x be her speed in meters per second. Then $36x = 12\pi$, and $x = \frac{\pi}{3}$.

13. **Answer (D):** Consider all ordered pairs (a, b) with each of the numbers a and b in the closed interval $[-20, 10]$. These pairs fill a 30×30 square in the coordinate plane, with an area of 900 square units. Ordered pairs in the first and third quadrants have the desired property, namely $a \cdot b > 0$. The areas of the portions of the 30×30 square in the first and third quadrants are $10^2 = 100$ and $20^2 = 400$, respectively. Therefore the probability of a positive product is $\frac{100+400}{900} = \frac{5}{9}$.

OR

Each of the numbers is positive with probability $\frac{1}{3}$ and negative with probability $\frac{2}{3}$. Their product is positive if and only if both numbers are positive or both are negative, so the requested probability is $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$.

14. **Answer (C):** Let x and y be the length and width of the parking lot, respectively. Then $xy = 168$ and $x^2 + y^2 = 25^2$. Note that

$$(x+y)^2 = x^2 + y^2 + 2xy = 25^2 + 2 \cdot 168 = 961.$$

Hence the perimeter is $2(x+y) = 2 \cdot \sqrt{961} = 62$.

Note that the dimensions of the parking lot are 7 and 24 meters.

15. **Answer (E):** If $x \neq 0$, then I is false:

$$x \,@\, (y+z) = \frac{x+(y+z)}{2} \neq \frac{x+y+x+z}{2} = \frac{x+y}{2} + \frac{x+z}{2} = (x \,@\, y) + (x \,@\, z).$$

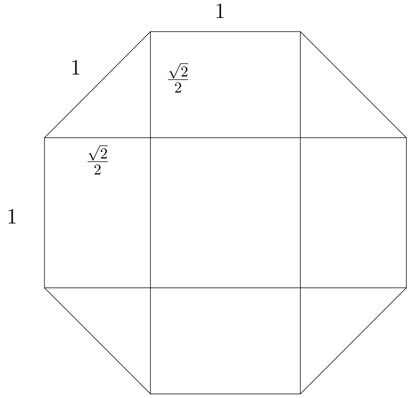
On the other hand, II and III are true for all values of x, y and z:

$$x + (y \,@\, z) = x + \frac{y+z}{2} = \frac{2x+y+z}{2} = \frac{(x+y)+(x+z)}{2} = (x+y) \,@\, (x+z),$$

and

$$x \,@\, (y \,@\, z) = \frac{x + \frac{y+z}{2}}{2} = \frac{\left(\frac{2x+y+z}{2}\right)}{2} = \frac{\frac{x+y}{2} + \frac{x+z}{2}}{2} = (x \,@\, y) \,@\, (x \,@\, z)$$

16. **Answer (A):** Assume the octagon's edge is 1. Then the corner triangles have hypotenuse 1 and thus legs $\frac{\sqrt{2}}{2}$ and area $\frac{1}{4}$ each; the four rectangles are 1 by $\frac{\sqrt{2}}{2}$ and have area $\frac{\sqrt{2}}{2}$ each, and the center square has area 1. The total area is $4 \cdot \frac{1}{4} + 4 \cdot \frac{\sqrt{2}}{2} + 1 = 2 + 2\sqrt{2}$. The probability that the dart hits the center square is $\frac{1}{2+2\sqrt{2}} = \frac{\sqrt{2}-1}{2}$.



17. **Answer (C):** Angle EAB is 90° because it subtends a diameter. Therefore angles BEA and ABE are 40° and 50°, respectively. Angle DEB is 50° because $\overline{AB}$ is parallel to $\overline{ED}$. Also, $\angle DEB$ is supplementary to $\angle CDE$, so $\angle CDE = 130°$. Because $\overline{EB}$ and $\overline{DC}$ are parallel chords, $ED = BC$ and $EBCD$ is an isosceles trapezoid. Thus $\angle BCD = \angle CDE = 130°$.

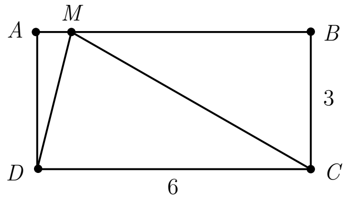
OR

Let O be the center of the circle. Establish, as in the first solution, that $\angle EAB = 90°$, $\angle BEA = 40°$, $\angle ABE = 50°$, and $\angle DEB = 50°$. Thus $\overline{AD}$ is a diameter and $\angle AOE = 100°$. By the Inscribed Angle Theorem

$$\angle BCD = \frac{1}{2}(\angle BOA + \angle AOE + \angle EOD) = \frac{1}{2}(80° + 100° + 80°) = 130°\,.$$

18. **Answer (E):** Sides $\overline{AB}$ and $\overline{CD}$ are parallel, so $\angle CDM = \angle AMD$. Because $\angle AMD = \angle CMD$, it follows that $\triangle CMD$ is isosceles and $CD = CM = 6$.

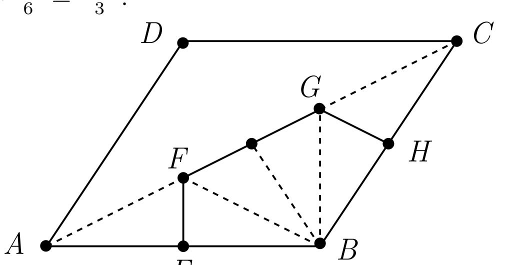
Therefore $\triangle MCB$ is a $30-60-90^\circ$ right triangle with $\angle BMC = 30^\circ$. Finally, $2 \cdot \angle AMD + 30^\circ = \angle AMD + \angle CMD + 30^\circ = 180^\circ$, so $\angle AMD = 75^\circ$.



19. **Answer (A):** The right side of the equation is defined only when $|x| \geq 4$. If $x \geq 4$, the equation is equivalent to $5x + 8 = x^2 - 16$, and the only solution with $x \geq 4$ is $x = 8$. If $x \leq -4$, the equation is equivalent to $8 - 5x = x^2 - 16$, and the only solution with $x \leq -4$ is $x = -8$. The product of the solutions is $-8 \cdot 8 = -64$.

20. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from the midpoint of $\overline{BC}$. Then the desired region R is the pentagon $BEFGH$.

Note that $\triangle AFE$ is a $30-60-90^\circ$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120^\circ - \angle FBE - \angle GBH = 60^\circ$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



21. **Answer (B):** The largest pairwise difference is 9, so $w - z = 9$. Let n be either x or y. Because n is between w and z,

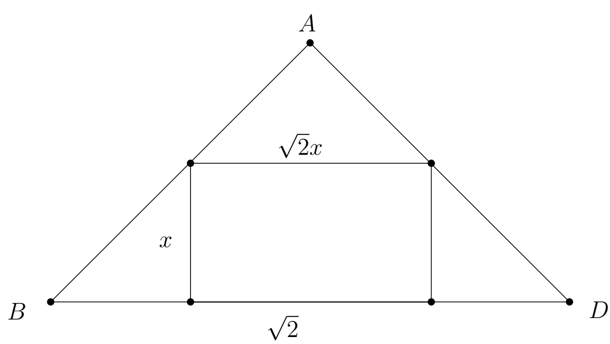
$$9 = w - z = (w - n) + (n - z).$$

Therefore the positive differences $w - n$ and $n - z$ must sum to 9. The given pairwise differences that sum to 9 are $3 + 6$ and $4 + 5$. The remaining pairwise difference must be $x - y = 1$.

The second largest pairwise difference is 6, so either $w - y = 6$ or $x - z = 6$. In the first case the set of four numbers may be expressed as $\{w, w-5, w-6, w-9\}$. Hence $4w - 20 = 44$, so $w = 16$. In the second case $w - x = 3$, and the four numbers may be expressed as $\{w, w-3, w-4, w-9\}$. Therefore $4w - 16 = 44$, so $w = 15$. The sum of the possible values for w is $16 + 15 = 31$.

Note: The possible sets of four numbers are $\{16, 11, 10, 7\}$ and $\{15, 12, 11, 6\}$.

22. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

$$(\sqrt{2} - 1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP

is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = (\frac{s}{2}, s)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

23. **Answer (D):** In the expansion of $(2000 + 11)^{2011}$, all terms except 11^{2011} are divisible by 1000, so the hundreds digit of 2011^{2011} is equal to that of 11^{2011}. Furthermore, in the expansion of $(10 + 1)^{2011}$, all terms except 1^{2011}, $\binom{2011}{1}(10)\left(1^{2010}\right)$, and $\binom{2011}{2}(10)^2\left(1^{2009}\right)$ are divisible by 1000. Thus the hundreds digit of 2011^{2011} is equal to that of

$$\begin{aligned} &1 + \binom{2011}{1}(10)\left(1^{2010}\right) + \binom{2011}{2}(10)^2\left(1^{2009}\right) \\ &= 1 + 2011 \cdot 10 + 2011 \cdot 1005 \cdot 100 \\ &= 1 + 2011 \cdot 100510. \end{aligned}$$

Finally, the hundreds digit of this number is equal to that of $1 + 11 \cdot 510 = 5611$, so the requested hundreds digit is 6.

24. **Answer (B):** For $0 < x \le 100$, the nearest lattice point directly above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0, 2)$ is $\frac{1}{2} + \frac{1}{x}$ if x is even and $\frac{1}{2} + \frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y = mx + 2$ contains no lattice point with $0 < x \le 100$ for $\frac{1}{2} < m < \frac{50}{99}$.

25. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$\begin{aligned} 2BE &= BE + BD = BE + CE + BD + AD - (AF + CF) \\ &= a + c - b, \end{aligned}$$

that is, $BE = \frac{1}{2}(a + c - b)$. Similarly $AD = \frac{1}{2}(b + c - a)$ and $CF = \frac{1}{2}(a + b - c)$. In the given $\triangle ABC$, suppose that $AB = x + 1$, $BC = x - 1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$\begin{aligned} BE &= \frac{1}{2}((x - 1) + (x + 1) - x) = \frac{1}{2}x, \\ AD &= \frac{1}{2}(x + (x + 1) - (x - 1)) = \frac{1}{2}x + 1, \text{ and} \\ CF &= \frac{1}{2}((x - 1) + x - (x + 1)) = \frac{1}{2}x - 1. \end{aligned}$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y-1, y, y+1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of T_{n+1} equals $\frac{1}{2}(a+b+c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011+2012+2013) = \frac{1509}{2^{N-3}}.$$

The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

$$-1+\frac{2012}{2^N}+\frac{2012}{2^N} \le 1+\frac{2012}{2^N}.$$

Equivalently, $2012 \le 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Davis, Steve Dunbar, Doug Faires, Sister Josannae Furey, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Joe Kennedy, Eugene Veklerov, David Wells, LeRoy Wenstrom, and Ron Yannone.

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