MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

13th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 7, 2012

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 10 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score 120 or above or finish in the top 2.5% on this AMC 10 will be invited to take the 30th annual American Invitational Mathematics Examination (AIME) on Thursday, March 15, 2012 or Wednesday, March 28, 2012. More details about the AIME and other information are on the back page of this test booklet.

2012

AMC 10 A

DO NOT OPEN UNTIL TUESDAY, February 7, 2012

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. PLEASE READ THE MANUAL BEFORE FEBRUARY 7, 2012. Nothing is needed from inside this package until February 7.
2. Your PRINCIPAL or VICE-PRINCIPAL must verify on the AMC 10 CERTIFICATION FORM (found in the Teachers' Manual) that you followed all rules associated with the conduct of the exam.
3. The Answer Forms must be mailed by trackable mail to the AMC office no later than 24 hours following the exam.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

The American Mathematics Competitions

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1. Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

   (A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

2. A square with side length 8 is cut in half, creating two congruent rectangles. What are the dimensions of one of these rectangles?

   (A) 2 by 4 **(B)** 2 by 6 **(C)** 2 by 8 **(D)** 4 by 4 **(E)** 4 by 8

3. A bug crawls along a number line, starting at -2. It crawls to -6, then turns around and crawls to 5. How many units does the bug crawl altogether?

   (A) 9 **(B)** 11 **(C)** 13 **(D)** 14 **(E)** 15

4. Let $\angle ABC = 24^\circ$ and $\angle ABD = 20^\circ$. What is the smallest possible degree measure for $\angle CBD$?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 12

5. Last year 100 adult cats, half of whom were female, were brought into the Smallville Animal Shelter. Half of the adult female cats were accompanied by a litter of kittens. The average number of kittens per litter was 4. What was the total number of cats and kittens received by the shelter last year?

   (A) 150 **(B)** 200 **(C)** 250 **(D)** 300 **(E)** 400

6. The product of two positive numbers is 9. The reciprocal of one of these numbers is 4 times the reciprocal of the other number. What is the sum of the two numbers?

   (A) $\frac{10}{3}$ **(B)** $\frac{20}{3}$ **(C)** 7 **(D)** $\frac{15}{2}$ **(E)** 8

7. In a bag of marbles, $\frac{3}{5}$ of the marbles are blue and the rest are red. If the number of red marbles is doubled and the number of blue marbles stays the same, what fraction of the marbles will be red?

   (A) $\frac{2}{5}$ **(B)** $\frac{3}{7}$ **(C)** $\frac{4}{7}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{4}{5}$

8. The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

9. A pair of six-sided fair dice are labeled so that one die has only even numbers (two each of 2, 4, and 6), and the other die has only odd numbers (two each of 1, 3, and 5). The pair of dice is rolled. What is the probability that the sum of the numbers on the tops of the two dice is 7?

   (A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

10. Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

    (A) 5 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

11. Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is BC?

    (A) 4 **(B)** 4.8 **(C)** 10.2 **(D)** 12 **(E)** 14.4

12. A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not by 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?

    (A) Friday **(B)** Saturday **(C)** Sunday **(D)** Monday **(E)** Tuesday

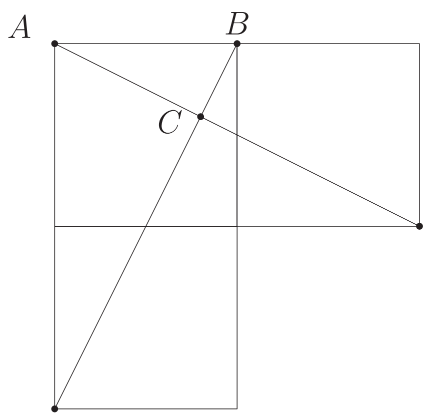
13. An *iterative average* of the numbers 1, 2, 3, 4, and 5 is computed in the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

    (A) $\frac{31}{16}$ **(B)** 2 **(C)** $\frac{17}{8}$ **(D)** 3 **(E)** $\frac{65}{16}$

14. Chubby makes nonstandard checkerboards that have 31 squares on each side. The checkerboards have a black square in every corner and alternate red and black squares along every row and column. How many black squares are there on such a checkerboard?

(A) 480 **(B)** 481 **(C)** 482 **(D)** 483 **(E)** 484

15. Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of $\triangle ABC$?



(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{\sqrt{2}}{4}$

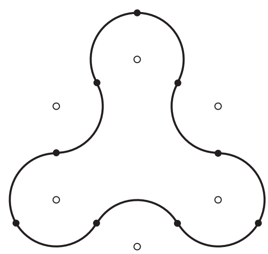
16. Three runners start running simultaneously from the same point on a 500-meter circular track. They each run clockwise around the course maintaining constant speeds of 4.4, 4.8, and 5.0 meters per second. The runners stop once they are all together again somewhere on the circular course. How many seconds do the runners run?

(A) 1,000 **(B)** 1,250 **(C)** 2,500 **(D)** 5,000 **(E)** 10,000

17. Let a and b be relatively prime integers with $a > b > 0$ and $\frac{a^3-b^3}{(a-b)^3} = \frac{73}{3}$. What is $a - b$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

18. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



(A) $2\pi + 6$ **(B)** $2\pi + 4\sqrt{3}$ **(C)** $3\pi + 4$ **(D)** $2\pi + 3\sqrt{3} + 2$ **(E)** $\pi + 6\sqrt{3}$

19. Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM, and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 PM. How long, in minutes, was each day's lunch break?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

20. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?

(A) $\dfrac{49}{512}$ **(B)** $\dfrac{7}{64}$ **(C)** $\dfrac{121}{1024}$ **(D)** $\dfrac{81}{512}$ **(E)** $\dfrac{9}{32}$

21. Let points $A = (0,0,0)$, $B = (1,0,0)$, $C = (0,2,0)$, and $D = (0,0,3)$. Points E, F, G, and H are midpoints of line segments $\overline{BD}$, $\overline{AB}$, $\overline{AC}$, and $\overline{DC}$ respectively. What is the area of $EFGH$?

(A) $\sqrt{2}$ **(B)** $\dfrac{2\sqrt{5}}{3}$ **(C)** $\dfrac{3\sqrt{5}}{4}$ **(D)** $\sqrt{3}$ **(E)** $\dfrac{2\sqrt{7}}{3}$

22. The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n?

(A) 255 **(B)** 256 **(C)** 257 **(D)** 258 **(E)** 259

23. Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

(A) 60 **(B)** 170 **(C)** 290 **(D)** 320 **(E)** 660

24. Let a, b, and c be positive integers with $a \geq b \geq c$ such that

$$\begin{aligned} a^2 - b^2 - c^2 + ab &= 2011 \text{ and} \\ a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc &= -1997. \end{aligned}$$

What is a?

(A) 249 **(B)** 250 **(C)** 251 **(D)** 252 **(E)** 253

25. Real numbers x, y, and z are chosen independently and at random from the interval $[0, n]$ for some positive integer n. The probability that no two of x, y, and z are within 1 unit of each other is greater than $\frac{1}{2}$. What is the smallest possible value of n?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

MATHEMATICAL ASSOCIATION OF AMERICA

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WRITE TO US!

Correspondence about the problems and solutions for this AMC 10
and orders for publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

2012 AIME

The 30th annual AIME will be held on Thursday, March 15, with the alternate on Wednesday, March 28. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above or finish in the top 2.5% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the 41st Annual USA Mathematical Olympiad (USAMO) on April 24-25, 2012. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: amc.maa.org