MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

13th Annual

AMC 10 B

American Mathematics Contest 10 B

Wednesday, February 22, 2012

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** There are $18-2=16$ more students than rabbits per classroom. Altogether there are $4\cdot 16=64$ more students than rabbits.

2. **Answer (E):** The width of the rectangle is the diameter of the circle, so the width is $2\cdot 5=10$. The length of the rectangle is $2\cdot 10=20$. Therefore the area of the rectangle is $10\cdot 20=200$.

3. **Answer (B):** The given point is 12 units above the horizontal line $y=2000$. The reflected point will be 12 units below the line, and 24 units below the given point. The coordinates of the reflected point are $(1000,1988)$.

4. **Answer (A):** The 7 marbles left over will fill one more bag of 6 marbles leaving 1 marble left over.

5. **Answer (D):** Tax is 10% and tip is 15%, so her total cost is $100\%+10\%+15\%=125\%$ of her meal. Thus her meal costs $\frac{\$27.50}{1.25}=\$22$.

6. **Answer (A):** Consider x and y as points on the real number line, with x necessarily to the right of y. Then $x-y$ is the distance between x and y. Xiaoli's rounding moved x to the right and moved y to the left. Therefore the distance between them increased, and her estimate is larger than $x-y$.

   To see that the other answer choices are not correct, let $x=2.9$ and $y=2.1$, and round each by 0.1. Then $x-y=0.8$ and Xiaoli's estimated difference is $(2.9+0.1)-(2.1-0.1)=1.0$.

7. **Answer (D):** Let h be the number of holes dug by the chipmunk. Then the chipmunk hid $3h$ acorns, while the squirrel hid $4(h-4)$ acorns. Since they hid the same number of acorns, $3h=4(h-4)$. Solving gives $h=16$. Thus the chipmunk hid $3\cdot 16=48$ acorns.

8. **Answer (B):** If $x-2>0$, then the given inequality is equivalent to $1<x-2<5$, or $3<x<7$. The integer solutions in this case are 4, 5, and 6. If $x-2<0$, then the given inequality is equivalent to $-5<x-2<-1$, or $-3<x<1$. The integer solutions in this case are -2, -1, and 0. The sum of all integer solutions is 12.

   OR

The given inequality is equivalent to $1 < |x - 2| < 5$. The solution set consists of all numbers whose distance from 2 on the number line is strictly between 1 and 5. Because only integer solutions are sought, this set is $\{-2, -1, 0, 4, 5, 6\}$. The required sum is 12.

9. **Answer (A):** The sum of two integers is even if they are both even or both odd. The sum of two integers is odd if one is even and one is odd. Only the middle two integers have an odd sum, namely $41 - 26 = 15$. Hence at least one integer must be even. A list satisfying the given conditions in which there is only one even integer is 1, 25, 1, 14, 1, 15.

10. **Answer (D):** Multiplying the given equation by $6N$ gives $MN = 36$. The divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36. Each of these divisors can be paired with a divisor to make a product of 36. Hence there are 9 ordered pairs (M, N).

11. **Answer (A):** There are 3 choices for Saturday (anything except cake) and for the same reason 3 choices for Thursday. Similarly there are 3 choices for Wednesday, Tuesday, Monday, and Sunday (anything except what was to be served the following day). Therefore there are $3^6 = 729$ possible dessert menus.

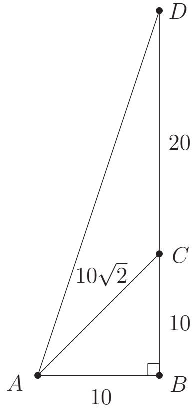
    OR

    If any dessert could be served on Friday, there would be 4 choices for Sunday and 3 for each of the other six days. There would be a total of $4 \cdot 3^6$ dessert menus for the week, and each dessert would be served on Friday with equal frequency. Because cake is the dessert for Friday, this total is too large by a factor of 4. The actual total is $3^6 = 729$.

12. **Answer (B):** Note that $\angle ABC = 90°$, so $\triangle ABC$ is a $45-45-90°$ triangle. Because hypotenuse $AC = 10\sqrt{2}$, the legs of $\triangle ABC$ have length 10. Therefore $AB = 10$ and $BD = BC + CD = 10 + 20 = 30$. By the Pythagorean Theorem,

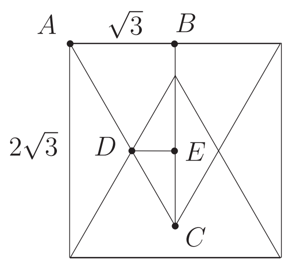
    $$AD = \sqrt{10^2 + 30^2} = \sqrt{1000}.$$

    Because $31^2 = 961$ and $32^2 = 1024$, it follows that $31 < AD < 32$.



13. **Answer (B):** Let x be Clea's rate of walking and r be the rate of the moving escalator. Because the distance is constant, $24(x+r) = 60x$. Solving for r yields $r = \frac{3}{2}x$. Let t be the time required for Clea to make the escalator trip while just standing on it. Then $rt = 60x$, so $\frac{3}{2}xt = 60x$. Therefore $t = 40$ seconds.

14. **Answer (D):** Construct the altitude for one of the equilateral triangles to its base on the square. Label the vertices of one of the resulting $30-60-90^\circ$ triangles A, B, and C, as shown. Then $AB = \sqrt{3}$ and $BC = 3$. Label one of the intersection points of the two equilateral triangles D and the center of the square E. Then $\triangle CDE$ is a $30-60-90^\circ$ triangle, $CE = 3 - \sqrt{3}$, and $DE = \frac{3-\sqrt{3}}{\sqrt{3}} = \sqrt{3} - 1$. The area of $\triangle CDE$ is $\frac{1}{2} \cdot (3 - \sqrt{3}) \cdot (\sqrt{3} - 1) = 2\sqrt{3} - 3$. Hence the area of the rhombus is $4 \cdot (2\sqrt{3} - 3) = 8\sqrt{3} - 12$.



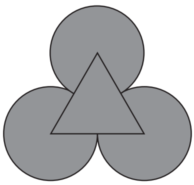
15. **Answer (D):** A total of 15 games are played, so all 6 teams could not be tied for the most wins as this would require $\frac{15}{6} = 2.5$ wins per team. However, it is possible for 5 teams to be tied, each with 3 wins and 2 losses. One such

outcome can be constructed by labeling 5 of the teams A, B, C, D, and E, and placing these labels at distinct points on a circle. If each of these teams beat the 2 labeled teams clockwise from its respective labeled point, as well as the remaining unlabeled team, all 5 would tie with 3 wins and 2 losses.

16. **Answer (A):** Connect the centers of the three circles to form an equilateral triangle with side length 4. Then the requested area is equal to the area of this triangle and a 300° sector of each circle. The equilateral triangle has base 4 and altitude $2\sqrt{3}$, so its area is

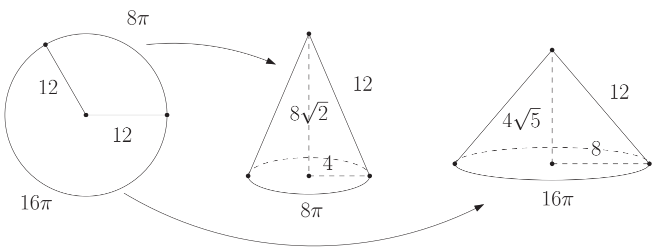
$$\frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3}.$$

The area of each sector is $\frac{300}{360} \cdot \pi \cdot 2^2 = \frac{10\pi}{3}$. Hence the total area is $3 \cdot \frac{10\pi}{3} + 4\sqrt{3} = 10\pi + 4\sqrt{3}$.



17. **Answer (C):** Each sector forms a cone with slant height 12. The circumference of the base of the smaller cone is $\frac{120}{360} \cdot 2 \cdot 12 \cdot \pi = 8\pi$. Hence the radius of the base of the smaller cone is 4 and its height is $\sqrt{12^2 - 4^2} = 8\sqrt{2}$. Similarly, the circumference of the base of the larger cone is 16π. Hence the radius of the base of the larger cone is 8 and its height is $4\sqrt{5}$. The ratio of the volume of the smaller cone to the volume of larger cone is

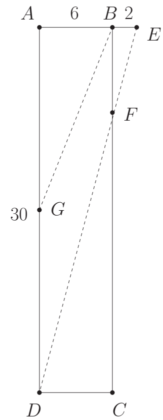
$$\frac{\frac{1}{3}\pi \cdot 4^2 \cdot 8\sqrt{2}}{\frac{1}{3}\pi \cdot 8^2 \cdot 4\sqrt{5}} = \frac{\sqrt{10}}{10}.$$



18. **Answer (C):** On average for every 500 people tested, 1 will test positive because he or she has the disease, while $2\% \cdot 499 \approx 10$ will test positive even though they do not have the disease. In other words, of approximately 11 people who test positive, only 1 has the disease, so the probability is approximately $\frac{1}{11}$.

19. **Answer (C):** Because $\triangle EBF$ is similar to $\triangle EAD$, it follows that $\frac{BF}{AD} = \frac{BE}{AE}$, or $\frac{BF}{30} = \frac{2}{8}$, giving $BF = \frac{15}{2}$. The area of trapezoid $BFDG$ is

$$\frac{1}{2}h(b_1 + b_2) = \frac{1}{2} \cdot AB \cdot (BF + GD) = \frac{1}{2} \cdot 6 \cdot \left(\frac{15}{2} + 15\right) = \frac{135}{2}.$$



20. **Answer (A):** The smallest initial number for which Bernardo wins after one round is the smallest integer solution of $2n + 50 \geq 1000$, which is 475. The smallest initial number for which he wins after two rounds is the smallest integer solution of $2n + 50 \geq 475$, which is 213. Similarly, the smallest initial numbers for which he wins after three and four rounds are 82 and 16, respectively. There is no initial number for which Bernardo wins after more than four rounds. Thus $N = 16$, and the sum of the digits of N is 7.

21. **Answer (A):** Since 4 of the 6 segments have length a, some 3 of the points (call them A, B, and C) must form an equilateral triangle of side length a. The fourth point D must be a distance a from one of A, B, or C, and without loss of generality it can be assumed to be A. Thus D lies on a circle of radius a centered at A. The distance from D to one of the other 2 points (which can be assumed to be B) is $2a$, so $\overline{BD}$ is a diameter of this circle and therefore is the hypotenuse of right triangle DCB with legs of lengths a and b. Thus $b^2 = (2a)^2 - a^2 = 3a^2$, and the ratio of b to a is $\sqrt{3}$.

22. **Answer (B):** If $a_1 = 1$, then the list must be an increasing sequence. Otherwise let $k = a_1$. Then the numbers 1 through $k-1$ must appear in increasing order from right to left, and the numbers from k through 10 must appear in increasing order from left to right. For $2 \le k \le 10$ there are $\binom{9}{k-1}$ ways to choose positions in the list for the numbers from 1 through $k-1$, and the positions of the remaining numbers are then determined. The number of lists is therefore

$$1 + \sum_{k=2}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512.$$

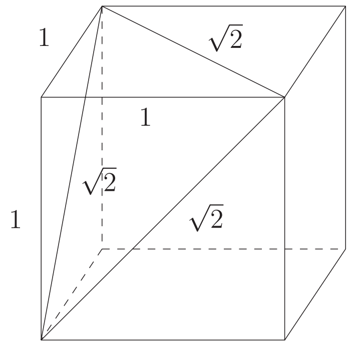
OR

If a_{10} is not 1 or 10, then numbers larger than a_{10} must appear in reverse order in the list, and numbers smaller than a_{10} must appear in order. However, 1 and 10 cannot both appear first in the list, so the placement of either 1 or 10 would violate the given conditions. Hence $a_{10} = 1$ or 10. By similar reasoning, when reading the list from right to left the number that appears next must be the smallest or largest unused integer from 1 to 10. This gives 2 choices for each term until there is one number left. Hence there are $2^9 = 512$ choices.

23. **Answer (D):** The discarded tetrahedron can be viewed as having an isosceles right triangle of side 1 as its base, with an altitude of 1. Therefore its volume is $\frac{1}{6}$. It can also be viewed as having an equilateral triangle of side length $\sqrt{2}$ as its base, in which case its altitude h must satisfy

$$\frac{1}{3} \cdot \frac{\sqrt{3}}{4} \left(\sqrt{2}\right)^2 \cdot h = \frac{1}{6},$$

which implies that $h = \frac{\sqrt{3}}{3}$. The height of the remaining solid is the long diagonal of the cube minus h, which is $\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$.



24. **Answer (B):** There are two cases to consider.

Case 1
Each song is liked by two of the girls. Then one of the three pairs of girls likes one of the six possible pairs of songs, one of the remaining pairs of girls likes one of the remaining two songs, and the last pair of girls likes the last song. This case can occur in $3 \cdot 6 \cdot 2 = 36$ ways.

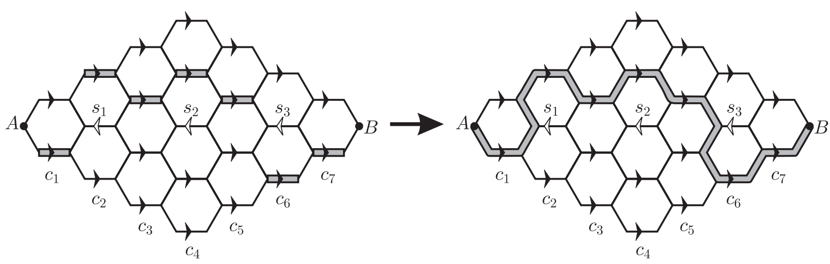
Case 2
Three songs are each liked by a different pair of girls, and the fourth song is liked by at most one girl. There are $4! = 24$ ways to assign the songs to these four categories, and the last song can be liked by Amy, Beth, Jo, or no one. This case can occur in $24 \cdot 4 = 96$ ways.

The total number of possibilities is $96 + 36 = 132$.

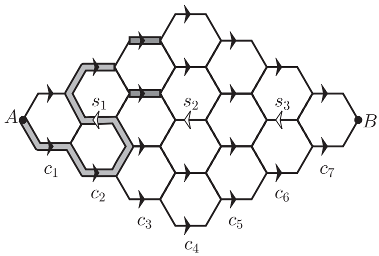
25. **Answer (E):**

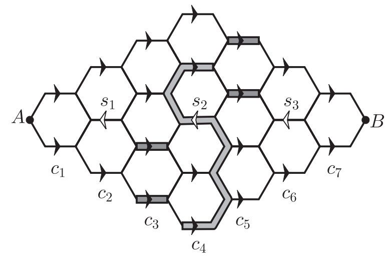
Label the columns having arrows as $c_1, c_2, c_3, \ldots, c_7$ according to the figure. Call those segments that can be traveled only from left to right *forward segments.* Call the segments s_1, s_2, and s_3, in columns c_2, c_4, and c_6, respectively, which can be traveled only from right to left, *back segments.* Denote S as the set of back segments traveled for a path.

First suppose that $S = \emptyset$. Because it is not possible to travel a segment more than once, it follows that the path is uniquely determined by choosing one forward segment in each of the columns c_j. There are 2, 2, 4, 4, 4, 2, and 2 choices for the forward segment in columns c_1, c_2, c_3, c_4, c_5, c_6, and c_7, respectively. This gives a total of 2^{10} total paths in this case.



Next suppose that $S = \{s_1\}$. The two forward segments in c_2, together with s_1, need to be part of the path, and once the forward segment from c_1 is chosen, the order in which the segments of c_2 are traveled is determined. Moreover, there are only 2 choices for possible segments in c_3 depending on the last segment traveled in c_2, either the bottom 2 or the top 2. For the rest of the columns, the path is determined by choosing any forward segment. Thus the total number of paths in this case is $2 \cdot 1 \cdot 2 \cdot 4 \cdot 4 \cdot 2 \cdot 2 = 2^8$, and by symmetry this is also the total for the number of paths when $S = \{s_3\}$. A similar argument gives $2 \cdot 1 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 2 = 2^6$ trips for the case when $S = \{s_1, s_3\}$.





Suppose $S = \{s_2\}$. Because s_2 is traveled, it follows that 2 forward segments in c_4 need to belong to the path, one of them above s_2 (2 choices) and the other below it (2 choices). Once these are determined, there are 2 possible choices for the order in which these segments are traveled: the bottom forward segment first, then s_2, then the top forward segment, or vice versa. Next, there are only 2 possible forward segments that can be selected in c_3 and also only 2 possible forward segments that can be selected in c_5. The forward segments in c_1, c_2, c_6, and c_7 can be freely selected (2 choices each). This gives a total of $(2^3 \cdot 2 \cdot 2) \cdot 2^4 = 2^9$ paths.

If $S = \{s_1, s_2\}$, then the analysis is similar, except for the last step, where the forward segments of c_1 and c_2 are determined by the previous choices. Thus there are $(2^3 \cdot 2 \cdot 2) \cdot 2^2 = 2^7$ possibilities, and by symmetry the same number when $S = \{s_2, s_3\}$.

Finally, if $S = \{s_1, s_2, s_3\}$, then in the last step, all forward segments of c_1, c_2, c_6, and c_7 are determined by the previous choices and hence there are $2^3 \cdot 2 \cdot 2 = 2^5$ possible paths. Altogether the total number of paths is $2^{10} + 2 \cdot 2^8 + 2^6 + 2^9 + 2 \cdot 2^7 + 2^5 = 2400$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Davis, Steve Dunbar, Doug Faires, Sister Josannae Furey, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Joe Kennedy, Eugene Veklerov, David Wells, LeRoy Wenstrom, and Ron Yannone.

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