1. **Answer (C):** Simplifying gives

$$\frac{2+4+6}{1+3+5} - \frac{1+3+5}{2+4+6} = \frac{12}{9} - \frac{9}{12} = \frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} = \frac{7}{12}\,.$$

2. **Answer (A):** The garden is $2 \cdot 15 = 30$ feet wide and $2 \cdot 20 = 40$ feet long. Hence Mr. Green expects $\frac{1}{2} \cdot 30 \cdot 40 = 600$ pounds of potatoes.

3. **Answer (C):** The difference between the high and low temperatures was 16 degrees, so the difference between each of these and the average temperature was 8 degrees. The low temperature was 8 degrees less than the average, so it was $3^\circ - 8^\circ = -5^\circ$.

4. **Answer (D):** The number 201 is the 1^{st} number counted when proceeding backwards from 201 to 3. In turn, 200 is the 2^{nd} number, 199 is the 3^{rd} number, and x is the $(202-x)^{\text{th}}$ number. Therefore 53 is the $(202-53)^{\text{th}}$ number, which is the 149^{th} number.

5. **Answer (B):** Note that $2 \cdot a - a \cdot b = (2-b)a$. This expression is negative when $b > 2$. Hence the product is minimized when a and b are as large as possible. The minimum value is $(2-5) \cdot 5 = -15$.

6. **Answer (C):** The sum of all the ages is $55 \cdot 33 + 33 \cdot 11 = 33 \cdot 66$, so the average of all the ages is

$$\frac{33 \cdot 66}{55+33} = \frac{33 \cdot 66}{88} = \frac{33 \cdot 3}{4} = 24.75\,.$$

7. **Answer (B):** The six points divide the circle into six arcs each measuring 60°. By the Inscribed Angle Theorem, the angles of the triangle can only be 30°, 60°, 90°, and 120°. Because the angles of the triangle are pairwise distinct the triangle must be a $30-60-90^\circ$ triangle. Therefore the hypotenuse of the triangle is the diameter of the circle, and the legs have lengths 1 and $\sqrt{3}$. The area of the triangle is $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$.

8. **Answer (B):** Let D equal the distance traveled by each car. Then Ray's car uses $\frac{D}{40}$ gallons of gasoline and Tom's car uses $\frac{D}{10}$ gallons of gasoline. The cars

combined miles per gallon of gasoline is

$$\frac{2D}{(\frac{D}{40} + \frac{D}{10})} = 16.$$

9. **Answer (D):** Note that

$$27{,}000 = 2^3 \cdot 3^3 \cdot 5^3.$$

The only three pairwise relatively prime positive integers greater than 1 with a product of 27,000 are 8, 27, and 125. The sum of these numbers is 160.

10. **Answer (C):** Let x denote the number of three-point shots attempted. Then the number of three-point shots made was $0.4x$, resulting in $3(0.4x) = 1.2x$ points. The number of two-point shots attempted was $1.5x$, and they were successful on $0.5(1.5x) = 0.75x$ of them resulting in $2(0.75x) = 1.5x$ points. The number of points scored was $1.2x + 1.5x = 54$, so $x = 20$.

11. **Answer (B):** By completing the square the equation can be rewritten as follows:

$$x^2 + y^2 = 10x - 6y - 34,$$
$$x^2 - 10x + 25 + y^2 + 6y + 9 = 0,$$
$$(x - 5)^2 + (y + 3)^2 = 0.$$

Therefore $x = 5$ and $y = -3$, so $x + y = 2$.

12. **Answer (B):** The five sides of the pentagon are congruent, and the five congruent diagonals are longer than the sides. Once one segment is selected, 4 of the 9 remaining segments have the same length as that segment. Therefore the requested probability is $\frac{4}{9}$.

13. **Answer (E):** Note that Jo starts by saying 1 number, and this is followed by Blair saying 2 numbers, then Jo saying 3 numbers, and so on. After someone completes her turn after saying the number n, then $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ numbers have been said. If $n = 9$ then 45 numbers have been said. Therefore there are $53 - 45 = 8$ more numbers that need to be said. The 53$^{\text{rd}}$ number said is 8.

14. **Answer (E):** The equation $x\clubsuit y = y\clubsuit x$ is equivalent to $x^2y-xy^2 = y^2x-yx^2$. This equation is euivalent to gives $2xy(x-y) = 0$. This equation will hold exactly if $x = 0$, $y = 0$, or $x = y$. The solution set consists of three lines: the x-axis, the y-axis, and the line $x = y$.

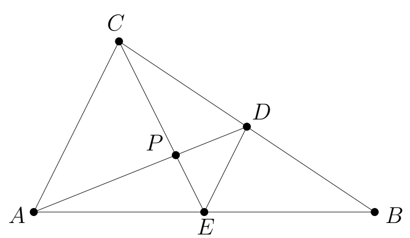
15. **Answer (B):** Let s be the side length of the triangle and h the side length of the hexagon. The hexagon can be subdivided into 6 equilateral triangles by drawing segments from the center of the hexagon to each vertex. Because the areas of the large triangle and hexagon are equal, the triangles in the hexagon each have area $\frac{1}{6}$ of the area of the large triangle. Thus

$$\frac{h}{s} = \sqrt{\frac{1}{6}} \qquad \text{so} \qquad h = \frac{\sqrt{6}}{6}s.$$

The perimeter of the triangle is $a = 3s$ and the perimeter of the hexagon is $b = 6h = \sqrt{6}s$, so

$$\frac{a}{b} = \frac{3s}{\sqrt{6}\,s} = \frac{\sqrt{6}}{2}.$$

16. **Answer (B):** The ratio of $PE{:}PD{:}DE$ is 3:4:5 .



Hence by the converse of the Pythagorean Theorem, $\triangle DPE$ is a right triangle. Therefore $\overline{CE}$ is perpendicular to $\overline{AD}$, and the area of $AEDC$ is one-half the product of its diagonals. Because P is the centroid of $\triangle ABC$, it follows that $CE = 3(PE) = 4.5$ and $AD = 3(PD) = 6$. Therefore the area of $AEDC$ is $0.5(4.5)(6) = 13.5$.

OR

From the first solution, triangles CPD, DPE, EPA, and APC are right triangles with right angle at P. The area of trapezoid $AEDC$ is given by the sum of the areas of these four triangles. Because $\overline{DE}$ is parallel to $\overline{AC}$ and D is the midpoint of $\overline{CB}$, triangles BAC and BED are similar with common ratio 2, so $AC = 2 \cdot DE = 5$. Triangles APC and DPE are similar, so $AP = 4$ and $CP = 3$. Thus the area of $AEDC$ is

$$\frac{1}{2} \cdot 4 \cdot 1.5 + \frac{1}{2} \cdot 3 \cdot 4 + \frac{1}{2} \cdot 2 \cdot 3 + \frac{1}{2} \cdot 2 \cdot 1.5 = 13.5\,.$$

17. **Answer (E):** After Alex makes m exchanges at the first booth and n exchanges at the second booth, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. No more exchanges are possible when he has fewer than 2 red tokens and fewer than 3 blue tokens. Therefore no more exchanges are possible if and only if $2m - n \geq 74$ and $3n - m \geq 73$. Equality can be achieved when $(m, n) = (59, 44)$, and Alex will have $59 + 44 = 103$ silver tokens.

    Note that the following exchanges produce 103 silver tokens:

	Red Tokens	Blue Tokens	Silver Tokens
Exchange 75 blue tokens	100	0	25
Exchange 100 red tokens	0	50	75
Exchange 48 blue tokens	16	2	91
Exchange 16 red tokens	0	10	99
Exchange 9 blue tokens	3	1	102
Exchange 2 red tokens	1	2	103

18. **Answer (D):** First note that the only number between 2000 and 2013 that shares this property is 2002.

    Consider now the numbers in the range 1001 to 1999. There is exactly 1 number, 1001, that shares the property when the units digits is 1. There are exactly 2 numbers, 1102 and 1012, when the units digit is 2; exactly 3 numbers, 1203, 1113, and 1023, when the units digits is 3, and so on. Because the thousands digit is always 1, when the units digit is n, for $1 \leq n \leq 9$, the sum of the hundreds and tens digits must be $n - 1$. There are exactly n ways for this to occur. Hence there are exactly

$$1 + (1 + 2 + \cdots + 9) = 1 + \tfrac{9 \cdot 10}{2} = 1 + 45 = 46$$

    numbers that share this property.

19. **Answer (D):** Let the common difference in the arithmetic sequence be d, so that $a = b + d$ and $c = b - d$. Because the quadratic has exactly one root, $b^2 - 4ac = 0$. Substitution gives $b^2 = 4(b + d)(b - d)$, and therefore $3b^2 = 4d^2$. Because $b \geq 0$ and $d \geq 0$, it follows that $\sqrt{3}b = 2d$. Thus the real root is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} = \frac{-b}{2(b+d)} = \frac{-b}{2\left(b + \frac{\sqrt{3}}{2}b\right)} = -2 + \sqrt{3}\,.$$

Note that the quadratic equation $x^2 + (4 - 2\sqrt{3})x + 7 - 4\sqrt{3}$ satisfies the given conditions.

20. **Answer (B):** The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. There must be a factor of 61 in the numerator, so $a_1 \geq 61$. Since $a_1!$ will have a factor of 59 and 2013 does not, there must be a factor of 59 in the denominator, and $b_1 \geq 59$. Thus $a_1 + b_1 \geq 120$, and this minimum value can be achieved only if $a_1 = 61$ and $b_1 = 59$. Furthermore, this minimum value is attainable because

$$2013 = \frac{(61!)(11!)(3!)}{(59!)(10!)(5!)}.$$

Thus $|a_1 - b_1| = a_1 - b_1 = 61 - 59 = 2$.

21. **Answer (C):** Let the two sequences be (a_n) and (b_n), and assume without loss of generality that $a_1 < b_1$. The definitions of the sequences imply that $a_7 = 5a_1 + 8a_2 = 5b_1 + 8b_2$, so $5(b_1 - a_1) = 8(a_2 - b_2)$. Because 5 and 8 are relatively prime, 8 divides $b_1 - a_1$ and 5 divides $a_2 - b_2$. It follows that $a_1 \leq b_1 - 8 \leq b_2 - 8 \leq a_2 - 13$. The minimum value of N results from choosing $a_1 = 0$, $b_1 = b_2 = 8$, and $a_2 = 13$, in which case $N = 104$.

22. **Answer (C):** The digit j at J contributes to all four sums, and each of the other digits contributes to exactly one sum. Therefore the sum of all four sums is $3j + (1 + 2 + 3 + \cdots + 9) = 45 + 3j$. Because all four sums are equal, this must be a multiple of 4, so $j = 1$, 5, or 9. For each choice of j, pair up the remaining digits so that each pair has the same sum. For example, for $j = 1$ the pairs are 2 and 9, 3 and 8, 4 and 7, and 5 and 6. Then order the pairs so that they correspond to the vertex pairs (A, E), (B, F), (C, G), (D, H). This results in $2^4 \cdot 4!$ different combinations for each j. Thus the requirements can be met in $2^4 \cdot 4! \cdot 3 = 1152$ ways.

23. **Answer (B):** The Pythagorean Theorem applied to right triangles ABD and ACD gives $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$; that is, $13^2 - BD^2 = 15^2 - (14 - BD)^2$, from which it follows that $BD = 5$, $CD = 9$, and $AD = 12$. Because triangles AED and ADC are similar,

$$\frac{AE}{12} = \frac{DE}{9} = \frac{12}{15},$$

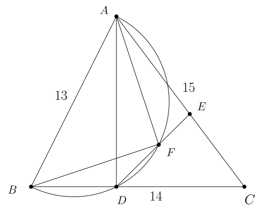
implying that $ED = \frac{36}{5}$ and $AE = \frac{48}{5}$.

Because $\angle AFB = \angle ADB = 90°$, it follows that $ABDF$ is cyclic. Thus $\angle ABD + \angle AFD = 180°$ from which $\angle ABD = \angle AFE$. Therefore right triangles ABD

and AFE are similar. Hence

$$\frac{FE}{5} = \frac{\frac{48}{5}}{12},$$

from which it follows that $FE = 4$. Consequently $DF = DE - FE = \frac{36}{5} - 4 = \frac{16}{5}$.



24. **Answer (A):** Let n denote a nice number from the given set. An integer m has exactly four divisors if and only if $m = p^3$ or $m = pq$, where p and q (with $p > q$) are prime numbers. In the former case, the sum of the four divisor is equal to $1 + p + p^2 + p^3$. Note that $1 + 11 + 11^2 + 11^3 < 2010 \le n$ and $1 + 13 + 13^2 + 13^3 > 2019 \ge n$. Therefore we must have $m = pq$ and $n = 1+q+p+pq = (1+q)(1+p)$. Because p is odd, n must be an even number. If $q = 2$, then n must be divisible by 3. In the given set only $2010 = (1+2)(1+669)$ and $2016 = (1 + 2)(1 + 671)$ satisfy these requirements. However neither 669 nor 671 are prime. If q is odd, then n must be divisible by 4. In the given set, only 2012 and 2016 are divisible by 4. None of the pairs of factors of 2012, namely $1 \cdot 2012$, $2 \cdot 1006$, $4 \cdot 503$, gives rise to primes p and q. This leaves $2016 = (1 + 3)(1 + 503)$, which is the only nice number in the given set.

Remark: Note that 2016 is nice in five ways. The other four ways are $(1+7)(1+251)$, $(1 + 11)(1 + 167)$, $(1 + 23)(1 + 83)$, and $(1 + 41)(1 + 47)$.

25. **Answer (E):** Expand the set of three-digit positive integers to include integers N, $0 \le N \le 99$, with leading zeros appended. Because $\mathrm{lcm}(5^2, 6^2, 10^2) = 900$, such an integer N meets the required condition if and only if $N + 900$ does. Therefore N can be considered to be chosen from the set of integers between 000 and 899, inclusive. Suppose that the last two digits in order of the base-5 representation of N are a_1 and a_0. Similarly, suppose that the last two digits of the base-6 representation of N are b_1 and b_0. By assumption, $2N \equiv a_0 + b_0 \pmod{10}$, but $N \equiv a_0 \pmod 5$ and so

$$a_0 + b_0 \equiv 2N \equiv 2a_0 \pmod{10}.$$

Thus $a_0 \equiv b_0 \pmod{10}$ and because $0 \le a_0 \le 4$ and $0 \le b_0 \le 5$, it follows that $a_0 = b_0$. Because $N \equiv a_0 \pmod 5$, it follows that there is an integer N_1 such that $N = 5N_1 + a_0$. Also, $N \equiv a_0 \pmod 6$ implies that $5N_1 + a_0 \equiv a_0$

(mod 6) and so $N_1 \equiv 0 \pmod 6$. It follows that $N_1 = 6N_2$ for some integer N_2 and so $N = 30N_2 + a_0$. Similarly, $N \equiv 5a_1 + a_0 \pmod{25}$ implies that $30N_2 + a_0 \equiv 5a_1 + a_0 \pmod{25}$ and then $N_2 \equiv 6N_2 \equiv a_1 \pmod 5$. It follows that $N_2 = 5N_3 + a_1$ for some integer N_3 and so $N = 150N_3 + 30a_1 + a_0$. Once more, $N \equiv 6b_1 + a_0 \pmod{36}$ implies that $6N_3 - 6a_1 + a_0 \equiv 150N_3 + 30a_1 + a_0 \equiv 6b_1 + a_0 \pmod{36}$ and then $N_3 \equiv a_1 + b_1 \pmod 6$. It follows that $N_3 = 6N_4 + a_1 + b_1$ for some integer N_4 and so $N = 900N_4 + 180a_1 + 150b_1 + a_0$. Finally, $2N \equiv 10(a_1 + b_1) + 2a_0 \pmod{100}$ implies that

$$60a_1 + 2a_0 \equiv 360a_1 + 300b_1 + 2a_0 \equiv 10a_1 + 10b_1 + 2a_0 \pmod{100}.$$

Therefore $5a_1 \equiv b_1 \pmod{10}$, equivalently, $b_1 \equiv 0 \pmod 5$ and $a_1 \equiv b_1 \pmod 2$. Conversely, if $N = 900N_4 + 180a_1 + 150b_1 + a_0$, $a_0 = b_0$, and $5a_1 \equiv b_1 \pmod{10}$, then $2N \equiv 60a_1 + 2a_0 = 10(a_1 + 5a_1) + a_0 + b_0 \equiv 10(a_1 + b_1) + (a_0 + b_0) \pmod{100}$. Because $0 \le a_1 \le 4$ and $0 \le b_1 \le 5$, it follows that there are exactly 5 different pairs (a_1, b_1), namely $(0,0)$, $(2,0)$, $(4,0)$, $(1,5)$, and $(3,5)$. Each of these can be combined with 5 different values of a_0 $(0 \le a_0 \le 4)$, to determine exactly 25 different numbers N with the required property.

The problems and solutions in this contest were proposed by Betsy Bennett, Steve Blasberg, Tom Butts, Steve Davis, Doug Faires, Zuming Feng, Michelle Ghrist, Jerry Grossman, Elgin Johnston, Jonathan Kane, Joe Kennedy, Cap Khoury, Dave Wells, and LeRoy Wenstrom.