MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

15th Annual

AMC 10 B

American Mathematics Contest 10 B

Wednesday, February 19, 2014

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** Leah has 7 pennies and 6 nickels, which are worth 37 cents.

2. **Answer (E):** Note that

$$\frac{2^3+2^3}{2^{-3}+2^{-3}} = \frac{2\cdot 2^3}{2\cdot 2^{-3}} = 2^6 = 64.$$

3. **Answer (E):** The fraction of Randy's trip driven on pavement was $1-\frac{1}{3}-\frac{1}{5} = \frac{7}{15}$. Therefore the entire trip was $20 \div \frac{7}{15} = \frac{300}{7}$ miles.

4. **Answer (B):** Let a muffin cost m dollars and a banana cost b dollars. Then $2(4m+3b) = 2m+16b$, and simplifying gives $m = \frac{5}{3}b$.

5. **Answer (A):** Denote the height of a pane by $5x$ and the width by $2x$. Then the square window has height $2\cdot 5x+6$ inches and width $4\cdot 2x+10$ inches. Solving $2\cdot 5x+6 = 4\cdot 2x+10$ gives $x=2$. The side length of the square window is 26 inches.

6. **Answer (C):** The special allows Orvin to purchase balloons at $\frac{1+\frac{2}{3}}{2} = \frac{5}{6}$ times the regular price. Because Orvin had just enough money to purchase 30 balloons at the regular price, he may now purchase $30\cdot\frac{6}{5} = 36$ balloons.

7. **Answer (A):** The fraction by which A is greater than B is simply the positive difference $A-B$ divided by B. The percent difference is 100 times this, or $100\left(\frac{A-B}{B}\right)$.

8. **Answer (E):** The truck travels for $3\cdot 60 = 180$ seconds, at a rate of $\frac{b}{6t}\cdot\frac{1}{3}$ yards per second. Hence the truck travels $180\cdot\frac{b}{6t}\cdot\frac{1}{3} = \frac{10b}{t}$ yards.

9. **Answer (A):** Note that

$$2014 = \frac{\frac{1}{w}+\frac{1}{z}}{\frac{1}{w}-\frac{1}{z}} = \frac{\frac{w+z}{wz}}{\frac{z-w}{wz}} = \frac{w+z}{z-w}.$$

Because $\frac{w+z}{z-w} = -\frac{w+z}{w-z}$, the requested value is -2014.

10. **Answer (C):** As indicated by the leftmost column $A+B \le 9$. Then both the second and fourth columns show that $C = 0$. Because A, B, and C are distinct digits, D must be at least 3. The following values for (A, B, C, D) show that D may be any of the 7 digits that are at least 3: $(1, 2, 0, 3)$, $(1, 3, 0, 4)$, $(2, 3, 0, 5)$, $(2, 4, 0, 6)$, $(2, 5, 0, 7)$, $(2, 6, 0, 8)$, $(2, 7, 0, 9)$.

11. **Answer (C):** If P is the price paid for an item, then the discounted prices with the three given discounts are given by the following calculations:

    (1) $(0.85)^2P = 0.7225P$ for a discount of 27.75%

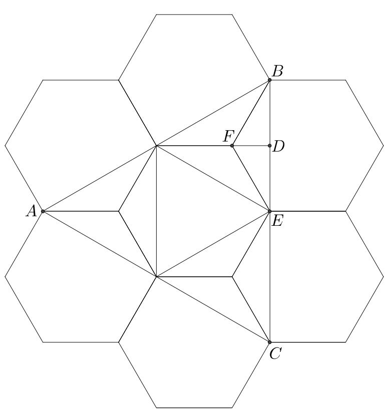
    (2) $(0.9)^3P = 0.729P$ for a discount of 27.1%

    (3) $(0.75) \cdot (0.95)P = 0.7125P$ for a discount of 28.75%

    The smallest integer greater than 27.75, 27.1, and 28.75 is 29.

12. **Answer (C):** By inspection, the five smallest positive divisors of 2,014,000,000 are 1, 2, 4, 5, and 8. Therefore the fifth largest divisor is $\frac{2{,}014{,}000{,}000}{8} = 251{,}750{,}000$.

13. **Answer (B):** Label points E and F as shown in the figure, and let D be the midpoint of $\overline{BE}$. Because $\triangle BFD$ is a $30-60-90^\circ$ triangle with hypotenuse 1, the length of $\overline{BD}$ is $\frac{\sqrt{3}}{2}$, and therefore $BC = 2\sqrt{3}$. It follows that the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4} \cdot (2\sqrt{3})^2 = 3\sqrt{3}$.



OR

Notice that $AE = 3$ since AE is composed of a hexagon side (length 1) and the longest diagonal of a hexagon (length 2). Triangle ABE is $30-60-90°$, so $BE = \frac{3}{\sqrt{3}} = \sqrt{3}$. The area of $\triangle ABC$ is $AE \cdot BE = 3\sqrt{3}$.

14. **Answer (D):** Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c-a)$, it is a multiple of 9. Therefore m is a multiple of 495. Because m is at most a 3-digit number and a is not equal to 0, $m = 495$. Therefore $c - a = 5$. Because $a + b + c \le 7$, the only possible abc is 106, so $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$.

OR

Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c-a)$, $c - a$ is a multiple of 5. Because $a \ge 1$ and $a+b+c \le 7$, it follows that $c = 6$ and $a = 1$. Therefore $b = 0$, so $a^2+b^2+c^2 = 37$.

15. **Answer (A):** Let $AD = \sqrt{3}$. Because $\angle ADE = 30°$, it follows that $AE = 1$ and $DE = 2$. Now $\angle EDF = 30°$ and $\angle DEF = 120°$, so $\triangle DEF$ is isosceles and $EF = 2$. Thus the area of $\triangle DEF$ (with $\overline{EF}$ viewed as the base) is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$, and the desired ratio is $\frac{\sqrt{3}}{\sqrt{3} \cdot 2\sqrt{3}} = \frac{\sqrt{3}}{6}$.

16. **Answer (B):** If exactly three of the four dice show the same number, then there are 6 possible choices for the repeated value and 5 possible choices for the non-repeated value. The non-repeated value may appear on any one of the 4 dice, so there are $6 \cdot 5 \cdot 4 = 120$ possible ways for such a result to occur. There are 6 ways for all four dice to show the same value. There are 6^4 total possible outcomes for the four dice. The probability of the desired result is $\frac{120+6}{6^4} = \frac{7}{72}$.

17. **Answer (D):** Note that

$$\begin{aligned}
10^{1002} - 4^{501} &= 2^{1002} \cdot 5^{1002} - 2^{1002} \\
&= 2^{1002}(5^{1002} - 1) \\
&= 2^{1002}(5^{501} - 1)(5^{501} + 1) \\
&= 2^{1002}(5-1)(5^{500} + 5^{499} + \cdots + 5 + 1)(5+1)(5^{500} - 5^{499} + \cdots \\
&\qquad - 5 + 1) \\
&= 2^{1005}(3)(5^{500} + 5^{499} + \cdots + 5 + 1)(5^{500} - 5^{499} + \cdots - 5 + 1).
\end{aligned}$$

Because each of the last two factors is a sum of an odd number of odd terms, they are both odd. The greatest power of 2 is 2^{1005}.

18. **Answer (E):** The numbers in the list have a sum of $11 \cdot 10 = 110$. The value of the 11th number is maximized when the sum of the first ten numbers is minimized subject to the following conditions.

    - If the numbers are arranged in nondecreasing order, the sixth number is 9.
    - The number 8 occurs either 2, 3, 4, or 5 times, and all other numbers occur fewer times.

    If 8 occurs 5 times, the smallest possible sum of the first 10 numbers is

    $$8 + 8 + 8 + 8 + 8 + 9 + 9 + 9 + 9 + 10 = 86.$$

    If 8 occurs 4 times, the smallest possible sum of the first 10 numbers is

    $$1 + 8 + 8 + 8 + 8 + 9 + 9 + 9 + 10 + 10 = 80.$$

    If 8 occurs 3 times, the smallest possible sum of the first 10 numbers is

    $$1 + 1 + 8 + 8 + 8 + 9 + 9 + 10 + 10 + 11 = 75.$$

    If 8 occurs 2 times, the smallest possible sum of the first 10 numbers is

    $$1 + 2 + 3 + 8 + 8 + 9 + 10 + 11 + 12 + 13 = 77.$$

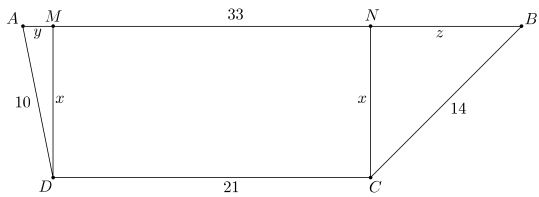
    Thus the largest possible value of the 11th number is $110 - 75 = 35$.

19. **Answer (D):** Let A be the first point chosen on the outer circle, let chords $\overline{AB}$ and $\overline{AC}$ on the outer circle be tangent to the inner circle at D and E, respectively, and let O be the common center of the two circles. Triangle ADO has a right angle at D, $OA = 2$, and $OD = 1$, so $\angle OAD = 30^\circ$. Similarly, $\angle OAE = 30^\circ$, so $\angle BAC = \angle DAE = 60^\circ$, and minor arc $BC = 120^\circ$. If X is the second point chosen on the outer circle, then chord $\overline{AX}$ intersects the inner circle if and only if X is on minor arc BC. Therefore the requested probability is $\frac{120^\circ}{360^\circ} = \frac{1}{3}$.

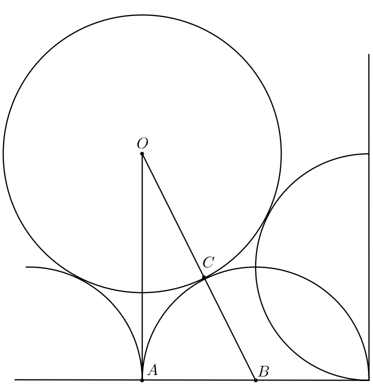
20. **Answer (C):** Note that $x^4 - 51x^2 + 50 = (x^2 - 50)(x^2 - 1)$, so the roots of the polynomial are ± 1 and $\pm\sqrt{50}$. Arranged from least to greatest, these roots are approximately $-7.1, -1, 1, 7.1$. The polynomial takes negative values on the intervals $(-7.1, -1)$ and $(1, 7.1)$, which include 12 integers: -7, -6, -5, -4, -3, -2, 2, 3, 4, 5, 6, 7.

21. **Answer (B):** Assume without loss of generality that $DA = 10$ and $BC = 14$. Let M and N be the feet of the perpendicular segments to $\overline{AB}$ from D and

C, respectively. The four points A, M, N, B appear on $\overline{AB}$ in that order. Let $x = DM = CN$, $y = AM$, and $z = NB$. Then $x^2 + y^2 = 10^2 = 100$, $x^2 + z^2 = 14^2 = 196$, and $y + 21 + z = 33$. Therefore $z = 12 - y$, and it follows that $\sqrt{196 - x^2} = 12 - \sqrt{100 - x^2}$. Squaring and simplifying gives $24\sqrt{100 - x^2} = 48$, so $x^2 = 96$ and $y = \sqrt{100 - 96} = 2$. The square of the length of the shorter diagonal, $\overline{AC}$, is $(y + 21)^2 + x^2 = 23^2 + 96 = 625$, so $AC = 25$.



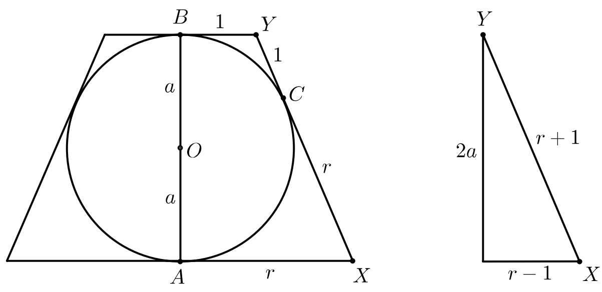
22. **Answer (B):** Let O be the center of the circle and choose one of the semicircles to have center point B. Label the point of tangency C and point A as in the figure. In $\triangle OAB$, $AB = \frac{1}{2}$ and $OA = 1$, so $OB = \frac{\sqrt{5}}{2}$. Because $BC = \frac{1}{2}$, $OC = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}$.



23. **Answer (E):** Assume without loss of generality that the radius of the top base of the truncated cone (frustum) is 1. Denote the radius of the bottom base by r and the radius of the sphere by a. The figure on the left is a side view of the frustum. Applying the Pythagorean Theorem to the triangle on the right yields $r = a^2$. The volume of the frustum is

$$\frac{1}{3}\pi(r^2 + r \cdot 1 + 1^2) \cdot 2a = \frac{1}{3}\pi(a^4 + a^2 + 1) \cdot 2a.$$

Setting this equal to twice the volume of the sphere, $\frac{4}{3}\pi a^3$, and simplifying gives $a^4 - 3a^2 + 1 = 0$, or $r^2 - 3r + 1 = 0$. Therefore $r = \frac{3+\sqrt{5}}{2}$.



24. **Answer (B):** The circular arrangement 14352 is bad because the sum 6 cannot be achieved with consecutive numbers, and the circular arrangement 23154 is bad because the sum 7 cannot be so achieved. It remains to show that these are the only bad arrangements. Given a circular arrangement, sums 1 through 5 can be achieved with a single number, and if the sum n can be achieved, then the sum $15 - n$ can be achieved using the complementary subset. Therefore an arrangement is not bad as long as sums 6 and 7 can be achieved. Suppose 6 cannot be achieved. Then 1 and 5 cannot be adjacent, so by a suitable rotation and/or reflection, the arrangement is $1bc5e$. Furthermore, $\{b, c\}$ cannot equal $\{2, 3\}$ because $1 + 2 + 3 = 6$; similarly $\{b, c\}$ cannot equal $\{2, 4\}$. It follows that $e = 2$, which then forces the arrangement to be 14352 in order to avoid consecutive 213. This arrangement is bad. Next suppose that 7 cannot be achieved. Then 2 and 5 cannot be adjacent, so again without loss of generality

the arrangement is $2bc5e$. Reasoning as before, $\{b,c\}$ cannot equal $\{3,4\}$ or $\{1,4\}$, so $e=4$, and then $b=3$ and $c=1$, to avoid consecutive 421; therefore the arrangement is 23154, which is also bad. Thus there are only two bad arrangements up to rotation and reflection.

25. **Answer (C):** First note that once the frog is on pad 5, it has probability $\frac{1}{2}$ of eventually being eaten by the snake, and a probability $\frac{1}{2}$ of eventually exiting the pond without being eaten. It is therefore necessary only to determine the probability that the frog on pad 1 will reach pad 5 before being eaten.

Consider the frog's jumps in pairs. The frog on pad 1 will advance to pad 3 with probability $\frac{9}{10}\cdot\frac{8}{10}=\frac{72}{100}$, will be back at pad 1 with probability $\frac{9}{10}\cdot\frac{2}{10}=\frac{18}{100}$, and will retreat to pad 0 and be eaten with probability $\frac{1}{10}$. Because the frog will eventually make it to pad 3 or make it to pad 0, the probability that it ultimately makes it to pad 3 is $\frac{72}{100}\div\left(\frac{72}{100}+\frac{10}{100}\right)=\frac{36}{41}$, and the probability that it ultimately makes it to pad 0 is $\frac{10}{100}\div\left(\frac{72}{100}+\frac{10}{100}\right)=\frac{5}{41}$.

Similarly, in a pair of jumps the frog will advance from pad 3 to pad 5 with probability $\frac{7}{10}\cdot\frac{6}{10}=\frac{42}{100}$, will be back at pad 3 with probability $\frac{7}{10}\cdot\frac{4}{10}+\frac{3}{10}\cdot\frac{8}{10}=\frac{52}{100}$, and will retreat to pad 1 with probability $\frac{3}{10}\cdot\frac{2}{10}=\frac{6}{100}$. Because the frog will ultimately make it to pad 5 or pad 1 from pad 3, the probability that it ultimately makes it to pad 5 is $\frac{42}{100}\div\left(\frac{42}{100}+\frac{6}{100}\right)=\frac{7}{8}$, and the probability that it ultimately makes it to pad 1 is $\frac{6}{100}\div\left(\frac{42}{100}+\frac{6}{100}\right)=\frac{1}{8}$.

The sequences of pairs of moves by which the frog will advance to pad 5 without being eaten are

$$1\to 3\to 5, 1\to 3\to 1\to 3\to 5, 1\to 3\to 1\to 3\to 1\to 3\to 5,$$

and so on. The sum of the respective probabilities of reaching pad 5 is then

$$\begin{aligned}
&\frac{36}{41}\cdot\frac{7}{8}+\frac{36}{41}\cdot\frac{1}{8}\cdot\frac{36}{41}\cdot\frac{7}{8}+\frac{36}{41}\cdot\frac{1}{8}\cdot\frac{36}{41}\cdot\frac{1}{8}\cdot\frac{36}{41}\cdot\frac{7}{8}+\cdots\\
&=\frac{63}{82}\left(1+\frac{9}{82}+\left(\frac{9}{82}\right)^2+\cdots\right)\\
&=\frac{63}{82}\div\left(1-\frac{9}{82}\right)\\
&=\frac{63}{73}.
\end{aligned}$$

Therefore the requested probability is $\frac{1}{2}\cdot\frac{63}{73}=\frac{63}{146}$.

OR

For $1\le j\le 5$, let p_j be the probability that the frog eventually reaches pad 10 starting at pad j. By symmetry $p_5=\frac{1}{2}$. For the frog to reach pad 10 starting from pad 4, the frog goes either to pad 3 with probability $\frac{2}{5}$ or to pad 5 with probability $\frac{3}{5}$, and then continues on a successful sequence from either of these pads. Thus $p_4 = \frac{2}{5}p_3 + \frac{3}{5}p_5 = \frac{2}{5}p_3 + \frac{3}{10}$. Similarly, to reach pad 10 starting from pad 3, the frog goes either to pad 2 with probability $\frac{3}{10}$ or to pad 4 with probability $\frac{7}{10}$. Thus $p_3 = \frac{3}{10}p_2 + \frac{7}{10}p_4$, and substituting from the previous equation for p_4 gives $p_3 = \frac{5}{12}p_2 + \frac{7}{24}$. In the same way, $p_2 = \frac{1}{5}p_1 + \frac{4}{5}p_3$ and after substituting for p_3 gives $p_2 = \frac{3}{10}p_1 + \frac{7}{20}$. Lastly, for the frog to escape starting from pad 1, it is necessary for it to get to pad 2 with probability $\frac{9}{10}$, and then escape starting from pad 2. Thus $p_1 = \frac{9}{10}p_2 = \frac{9}{10}(\frac{3}{10}p_1 + \frac{7}{20})$, and solving the equation gives $p_1 = \frac{63}{146}$.

Note: This type of random process is called a Markov process.

The problems and solutions in this contest were proposed by Steve Blasberg, Tom Butts, Peter Gilchrist, Jerry Grossman, Jon Kane, Joe Kennedy, Cap Khoury, Stuart Sidney, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.