MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

17th Annual

AMC 10A

American Mathematics Contest 10A

Tuesday, February 2, 2016

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (B):**

$$\frac{11!-10!}{9!}=\frac{10!\cdot(11-1)}{9!}=\frac{10\cdot 9!\cdot 10}{9!}=100$$

2. **Answer (C):** The equation can be written $10^x\cdot(10^2)^{2x}=(10^3)^5$ or $10^x\cdot 10^{4x}=10^{15}$. Thus $10^{5x}=10^{15}$, so $5x=15$ and $x=3$.

3. **Answer (C):** Because $\$12.50=50\cdot\$0.25$, Ben spent $\$50$. David spent $\$50-\$12.50=\$37.50$, and the two together paid $\$87.50$.

4. **Answer (B):**

$$\frac{3}{8}-\left(-\frac{2}{5}\right)\left\lfloor\frac{\frac{3}{8}}{-\frac{2}{5}}\right\rfloor=\frac{3}{8}+\frac{2}{5}\left\lfloor-\frac{15}{16}\right\rfloor=\frac{3}{8}+\frac{2}{5}(-1)=-\frac{1}{40}$$

5. **Answer (D):** Let the dimensions of the box be x, $3x$, and $4x$. Then the volume of the box is $12x^3$. Therefore the volume must be 12 times the cube of an integer. Among the choices, only $48=4\cdot 12$, $96=8\cdot 12$, and $144=12\cdot 12$ are multiples of 12, and only for 96 is the other factor a perfect cube.

6. **Answer (D):** Each time Emilio replaces a 2 in the ones position by 1, Ximena's sum is decreased by 1. When Emilio replaces a 2 in the tens position by 1, Ximena's sum is decreased by 10. Ximena wrote 3 twos in the ones position $(2,12,22)$ and 10 twos in the tens position $(20,21,22,\ldots,29)$. Thus Ximena's sum is greater than Emilio's sum by $3\cdot 1+10\cdot 10=103$.

7. **Answer (D):** The mean of the data values is

$$\frac{60+100+x+40+50+200+90}{7}=\frac{x+540}{7}=x.$$

Solving this equation for x gives $x=90$. Thus the data in nondecreasing order are $40,50,60,90,90,100,200$, so the median is 90 and the mode is 90, as required.

8. **Answer (C):** Working backwards, Fox must have approached the bridge for the third time with 20 coins in order to have no coins left after paying the

toll. In the second crossing he must have started with 30 coins in order to have $20 + 40 = 60$ before paying the toll. So he must have started with 35 coins in order to have $30 + 40 = 70$ before paying the toll for the first crossing.

OR

Let c be the number of coins Fox had at the beginning. After three crossings he had $2(2(2c - 40) - 40) - 40 = 8c - 280$ coins. Setting this equal to 0 and solving gives $c = 35$.

9. **Answer (D):** There are

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

coins in the array. Therefore $N(N+1) = 2 \cdot 2016 = 4032$. Because $N(N+1) \approx N^2$, it follows that $N \approx \sqrt{4032} \approx \sqrt{2^{12}} = 2^6 = 64$. Indeed, $63 \cdot 64 = 4032$, so $N = 63$ and the sum of the digits of N is 9.

10. **Answer (B):** Let the inner rectangle's length be x feet; then its area is x square feet. The middle region has area $3(x+2) - x = 2x + 6$, so the difference in the arithmetic sequence is equal to $(2x+6) - x = x + 6$. The outer region has area $5(x+4) - 3(x+2) = 2x + 14$, so the difference in the arithmetic sequence is also equal to $(2x + 14) - (2x + 6) = 8$. From $x + 6 = 8$, it follows that $x = 2$. The regions have areas 2, 10, and 18.

11. **Answer (D):** The diagonal of the rectangle from upper left to lower right divides the shaded region into four triangles. Two of them have a 1-unit horizontal base and altitude $\frac{1}{2} \cdot 5 = 2\frac{1}{2}$, and the other two have a 1-unit vertical base and altitude $\frac{1}{2} \cdot 8 = 4$. Therefore the total area is $2 \cdot \frac{1}{2} \cdot 1 \cdot 2\frac{1}{2} + 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 = 6\frac{1}{2}$.

12. **Answer (A):** The product of three integers is odd if and only if all three integers are odd. There are 1008 odd integers among the 2016 integers in the given range. The probability that all the selected integers are odd is

$$p = \frac{1008}{2016} \cdot \frac{1007}{2015} \cdot \frac{1006}{2014}.$$

The first factor is $\frac{1}{2}$ and each of the other factors is less than $\frac{1}{2}$, so $p < \frac{1}{8}$.

13. **Answer (B):** The total number of seats moved to the right among the five friends must equal the total number of seats moved to the left. One of Dee and

Edie moved some number of seats to the right, and the other moved the same number of seats to the left. Because Bea moved two seats to the right and Ceci moved one seat to the left, Ada must also move one seat to the left upon her return. Because her new seat is an end seat and its number cannot be 5, it must be seat 1. Therefore Ada occupied seat 2 before she got up. The order before moving was Bea-Ada-Ceci-Dee-Edie (or Bea-Ada-Ceci-Edie-Dee), and the order after moving was Ada-Ceci-Bea-Edie-Dee (or Ada-Ceci-Bea-Dee-Edie).

14. **Answer (C):** If the sum uses n twos and m threes, then $2n + 3m = 2016$. Therefore $n = \frac{2016-3m}{2}$. Both m and n will be nonnegative integers if and only if m is an even integer from 0 to 672. Thus there are $\frac{672}{2} + 1 = 337$ ways to form the sum.

15. **Answer (A):** The circle of dough has radius 3 inches. The area of the remaining dough is $3^2 \cdot \pi - 7\pi = 2\pi$ in^2. Let r be the radius in inches of the scrap cookie; then $2\pi = \pi r^2$. Therefore $r = \sqrt{2}$ inches.

16. **Answer (D):** After reflection about the x-axis, the coordinates of the image are $A'(0,-2)$, $B'(-3,-2)$, and $C'(-3,0)$. The counterclockwise $90°$-rotation around the origin maps this triangle to the triangle with vertices $A''(2,0)$, $B''(2,-3)$, and $C''(0,-3)$. Notice that the final image can be mapped to the original triangle by interchanging the x- and y-coordinates, which corresponds to a reflection about the line $y = x$.

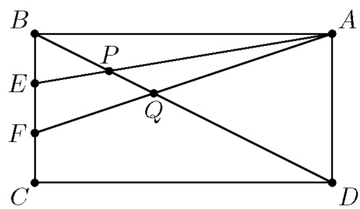
17. **Answer (A):** Let $N = 5k$, where k is a positive integer. There are $5k + 1$ equally likely possible positions for the red ball in the line of balls. Number these $0, 1, 2, 3, \ldots, 5k-1, 5k$ from one end. The red ball will *not* divide the green balls so that at least $\frac{3}{5}$ of them are on the same side if it is in position $2k+1, 2k+2, \ldots, 3k-1$. This includes $(3k-1) - 2k = k-1$ positions. The probability that $\frac{3}{5}$ or more of the green balls will be on the same side is therefore $1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}$.

    Solving the inequality $\frac{4k+2}{5k+1} < \frac{321}{400}$ for k yields $k > \frac{479}{5} = 95\frac{4}{5}$. The value of k corresponding to the required least value of N is therefore 96, so $N = 480$. The sum of the digits of N is 12.

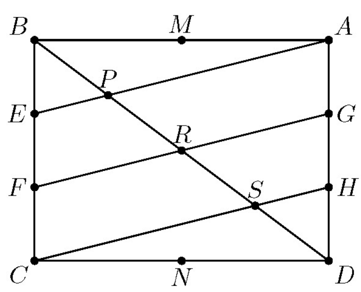
18. **Answer (C):** The sum of the four numbers on the vertices of each face must be $\frac{1}{6} \cdot 3 \cdot (1+2+\cdots+8) = 18$. The only sets of four of the numbers that include 1 and have a sum of 18 are $\{1,2,7,8\}$, $\{1,3,6,8\}$, $\{1,4,5,8\}$, and $\{1,4,6,7\}$. Three of these sets contain both 1 and 8. Because two specific vertices can

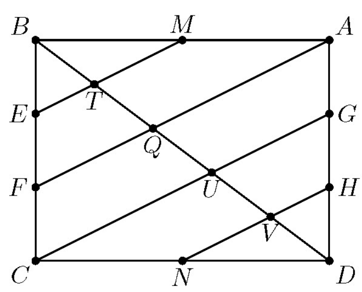
belong to at most two faces, the vertices of one face must be labeled with the numbers $1, 4, 6, 7$, and two of the faces must include vertices labeled 1 and 8. Thus 1 and 8 must mark two adjacent vertices. The cube can be rotated so that the vertex labeled 1 is at the lower left front, and the vertex labeled 8 is at the lower right front. The numbers 4, 6, and 7 must label vertices on the left face. There are $3! = 6$ ways to assign these three labels to the three remaining vertices of the left face. Then the numbers 5, 3, and 2 must label the vertices of the right face adjacent to the vertices labeled 4, 6, and 7, respectively. Hence there are 6 possible arrangements.

19. **Answer (E):** Triangles APD and EPB are similar and $BE : DA = 1 : 3$, so $BP = \frac{1}{4}BD$. Triangles AQD and FQB are similar and $BF : DA = 2 : 3$, so $BQ = \frac{2}{5}BD$ and $QD = \frac{3}{5}BD$. Then $PQ = BQ - BP = (\frac{2}{5} - \frac{1}{4})BD = \frac{3}{20}BD$. Thus $BP : PQ : QD = \frac{1}{4} : \frac{3}{20} : \frac{3}{5} = 5 : 3 : 12$, and $r + s + t = 5 + 3 + 12 = 20$.



Note: The answer is independent of the dimensions of the original rectangle. Consider the figures below, showing the rectangle $ABCD$ with points E and F trisecting side $\overline{BC}$. Let G and H trisect $\overline{AD}$, and let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$. Then the segments $\overline{AE}$, $\overline{GF}$, and $\overline{HC}$ are equally spaced, implying that $BP = PR = RS = SD$ and showing that $BP : PD : BD = 1 : 3 : 4 = 5 : 15 : 20$. The segments $\overline{ME}$, $\overline{AF}$, $\overline{GC}$, and $\overline{HN}$ are also equally spaced, implying that $BT = TQ = QU = UV = VD$ and showing that $BQ : QD : BD = 2 : 3 : 5 = 8 : 12 : 20$. It then follows that $BP : PQ : QD = 5 : (15 - 12) : 12 = 5 : 3 : 12$.





20. **Answer (B):** If a term contains all four variables a, b, c, and d, then it has the form $a^{i+1}b^{j+1}c^{k+1}d^{l+1}1^m$ for some nonnegative integers i, j, k, l, and m such that $(i+1)+(j+1)+(k+1)+(l+1)+m=N$ or $i+j+k+l+m=N-4$. The number of terms can be counted using the stars and bars technique. The number of linear arrangements of $N-4$ stars and 4 bars corresponds to the number of possible values of i, j, k, l, and m. Namely, in each arrangement the bars separate the stars into five groups (some of them can be empty) whose sizes are the values of i, j, k, l, and m. There are

$$\binom{N-4+4}{4}=\binom{N}{4}=\frac{N(N-1)(N-2)(N-3)}{4\cdot3\cdot2\cdot1}=1001=7\cdot11\cdot13$$

such arrangements. So $N(N-1)(N-2)(N-3)=4\cdot3\cdot2\cdot7\cdot11\cdot13=14\cdot13\cdot12\cdot11$. Thus the answer is $N=14$.

21. **Answer (D):** Let X be the foot of the perpendicular from P to $\overline{QQ'}$, and let Y be the foot of the perpendicular from Q to $\overline{RR'}$. By the Pythagorean Theorem,

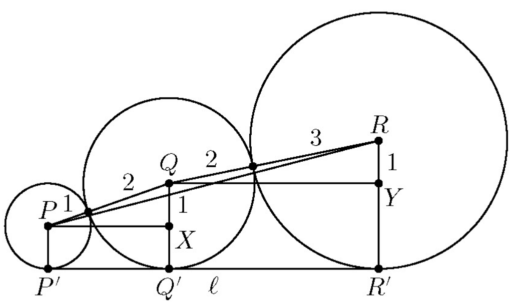
$$P'Q'=PX=\sqrt{(2+1)^2-(2-1)^2}=\sqrt{8}$$

and

$$Q'R'=QY=\sqrt{(3+2)^2-(3-2)^2}=\sqrt{24}.$$

The required area can be computed as the sum of the areas of the two smaller trapezoids, $PQQ'P'$ and $QRR'Q'$, minus the area of the large trapezoid, $PRR'P'$:

$$\frac{1+2}{2}\sqrt{8}+\frac{2+3}{2}\sqrt{24}-\frac{1+3}{2}\left(\sqrt{8}+\sqrt{24}\right)=\sqrt{6}-\sqrt{2}.$$



22. **Answer (D):** Let $110n^3=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where the p_j are distinct primes and the r_j are positive integers. Then $\tau(110n^3)$, the number of positive integer divisors of $110n^3$, is given by

$$\tau(110n^3)=(r_1+1)(r_2+1)\cdots(r_k+1)=110.$$

Because $110 = 2 \cdot 5 \cdot 11$, it follows that $k = 3$, $\{p_1, p_2, p_3\} = \{2, 5, 11\}$, and, without loss of generality, $r_1 = 1$, $r_2 = 4$, and $r_3 = 10$. Therefore

$$n^3 = \frac{p_1 \cdot p_2^4 \cdot p_3^{10}}{110} = p_2^3 \cdot p_3^9, \quad \text{so} \quad n = p_2 \cdot p_3^3.$$

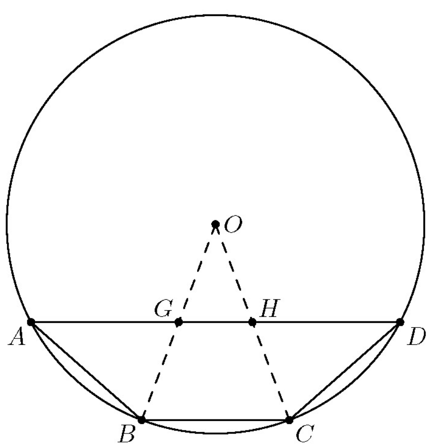
It follows that $81n^4 = 3^4 \cdot p_2^4 \cdot p_3^{12}$, and because 3, p_2, and p_3 are distinct primes, $\tau(81n^4) = 5 \cdot 5 \cdot 13 = 325$.

23. **Answer (A):** From the given properties, $a \diamondsuit 1 = a \diamondsuit (a \diamondsuit a) = (a \diamondsuit a) \cdot a = 1 \cdot a = a$ for all nonzero a. Then for nonzero a and b, $a = a \diamondsuit 1 = a \diamondsuit (b \diamondsuit b) = (a \diamondsuit b) \cdot b$. It follows that $a \diamondsuit b = \frac{a}{b}$. Thus

$$100 = 2016 \diamondsuit (6 \diamondsuit x) = 2016 \diamondsuit \frac{6}{x} = \frac{2016}{\frac{6}{x}} = 336x,$$

so $x = \frac{100}{336} = \frac{25}{84}$. The requested sum is $25 + 84 = 109$.

24. **Answer (E):** Let $ABCD$ be the given quadrilateral inscribed in the circle centered at O, with $AB = BC = CD = 200$, as shown in the figure. Because the chords $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are shorter than the radius, each of $\angle AOB$, $\angle BOC$, and $\angle COD$ is less than 60°, so O is outside the quadrilateral $ABCD$. Let G and H be the intersections of $\overline{AD}$ with $\overline{OB}$ and $\overline{OC}$, respectively. Because $\overline{AD}$ and $\overline{BC}$ are parallel, and $\triangle OAB$ and $\triangle OBC$ are congruent and isosceles, it follows that $\angle ABO = \angle OBC = \angle OGH = \angle AGB$. Thus $\triangle ABG$, $\triangle OGH$, and $\triangle OBC$ are similar and isosceles with $\frac{AB}{BG} = \frac{OG}{GH} = \frac{OB}{BC} = \frac{200\sqrt{2}}{200} = \sqrt{2}$. Then $AG = AB = 200$, $BG = \frac{AB}{\sqrt{2}} = \frac{200}{\sqrt{2}} = 100\sqrt{2}$, and $GH = \frac{OG}{\sqrt{2}} = \frac{BO-BG}{\sqrt{2}} = \frac{200\sqrt{2}-100\sqrt{2}}{\sqrt{2}} = 100$. Therefore $AD = AG + GH + HD = 200 + 100 + 200 = 500$.



25. **Answer (A):** Because $\text{lcm}(x, y) = 2^3 \cdot 3^2$ and $\text{lcm}(x, z) = 2^3 \cdot 3 \cdot 5^2$, it follows that 5^2 divides z, but neither x nor y is divisible by 5. Furthermore, y is divisible by 3^2, and neither x nor z is divisible by 3^2, but at least one of x or z is divisible by 3. Finally, because $\text{lcm}(y, z) = 2^2 \cdot 3^2 \cdot 5^2$, at least one of y or z is divisible by 2^2, but neither is divisible by 2^3. However, x must be divisible by 2^3. Thus $x = 2^3 \cdot 3^j$, $y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, where $\max(j, n) = 1$ and $\max(k, m) = 2$. There are 3 choices for (j, n) and 5 choices for (k, m), so there are 15 possible ordered triples (x, y, z).

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