MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

17th Annual

AMC 10B

American Mathematics Contest 10B

Wednesday, February 17, 2016

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (D):**

$$\frac{2(\frac{1}{2})^{-1} + \frac{(\frac{1}{2})^{-1}}{2}}{\frac{1}{2}} = \left(2 \cdot 2 + \frac{2}{2}\right) \cdot 2 = 10$$

2. **Answer (B):**

$$\frac{2\heartsuit 4}{4\heartsuit 2} = \frac{2^3 \cdot 4^2}{4^3 \cdot 2^2} = \frac{2}{4} = \frac{1}{2}$$

3. **Answer (D):**

$$\Bigg|\Big|\,|-2016|-(-2016)\,\Big|-|-2016|\Bigg|-(-2016)$$
$$= \Big|\,\big|\,2016+2016\,\big|-2016\,\Big|+2016 = 2016+2016 = 4032$$

4. **Answer (B):** It took Zoey $1+2+3+\cdots+15 = \frac{15\cdot 16}{2} = 120$ days to read the 15 books. Because $120 = 7 \cdot 17 + 1$, it follows that Zoey finished the 15th book on the same day of the week as the first, a Monday.

5. **Answer (D):** Because the mean is 8, it follows that the sum of the ages of all Amanda's cousins is $8 \cdot 4 = 32$. Because the median age is 5, the sum of the two middle ages is $5 \cdot 2 = 10$. Then the sum of the ages of Amanda's youngest and oldest cousins is $32 - 10 = 22$.

6. **Answer (B):** Because S has to be greater than 300, the digit sum has to be at least 4, and an example like $197 + 203 = 400$ shows that 4 is indeed the smallest possible value.

7. **Answer (C):** Let α and β be the measures of the angles, with $\alpha < \beta$. Then $\frac{\beta}{\alpha} = \frac{5}{4}$. Because $\alpha < \beta$, it follows that $90^\circ - \beta < 90^\circ - \alpha$, so $90^\circ - \alpha = 2(90^\circ - \beta)$. This leads to the system of linear equations $4\beta - 5\alpha = 0$ and $2\beta - \alpha = 90^\circ$. Solving the system gives $\alpha = 60^\circ$, $\beta = 75^\circ$. The requested sum is $\alpha + \beta = 135^\circ$.

8. **Answer (A):** Positive even powers of numbers ending in 5 end in 25. The tens digit of the difference is the tens digit of $25 - 17 = 08$, or 0.

9. **Answer (C):** Let the vertex of the triangle that lies in the first quadrant be (x, x^2). Then the base of the triangle is $2x$ and the height is x^2, so $\frac{1}{2} \cdot 2x \cdot x^2 = 64$. Thus $x^3 = 64$, $x = 4$, and $BC = 2x = 8$.

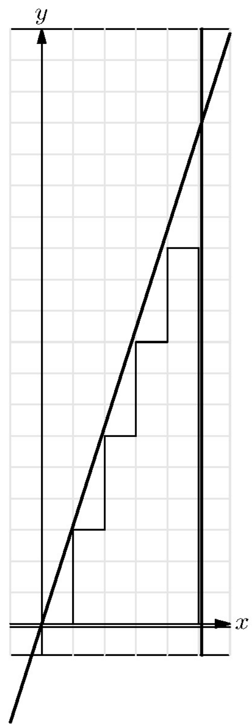
10. **Answer (D):** The weight of an object of uniform density is proportional to its volume. The volume of the triangular piece of wood of uniform thickness is proportional to the area of the triangle. The side length of the second piece is $\frac{5}{3}$ times the side length of the first piece, so the area of the second piece is $\left(\frac{5}{3}\right)^2$ times the area of the first piece. Therefore the weight is $12 \cdot \left(\frac{5}{3}\right)^2 = \frac{100}{3} \approx 33.3$ ounces.

11. **Answer (B):** Let x be the number of posts along the shorter side; then there are $2x$ posts along the longer side. When counting the number of posts on all the sides of the garden, each corner post is counted twice, so $2x + 2(2x) = 20 + 4$. Solving this equation gives $x = 4$. Thus the dimensions of the rectangle are $(4 - 1) \cdot 4 = 12$ yards by $(8 - 1) \cdot 4 = 28$ yards. The requested area is given by the product of these dimensions, $12 \cdot 28 = 336$ square yards.

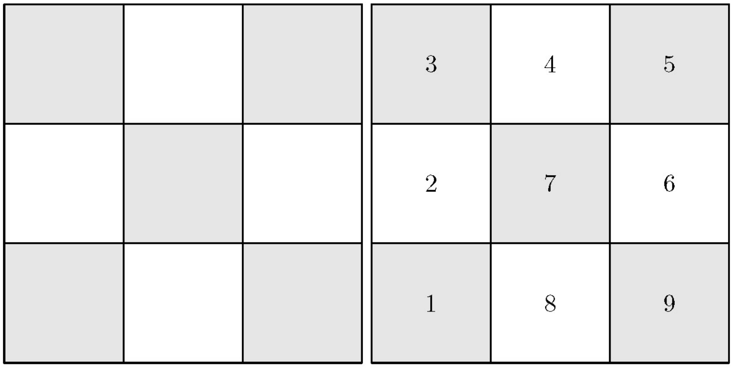
12. **Answer (D):** The product of two integers is odd if and only if both integers are odd. Thus the probability that the product is odd is $\frac{3}{5} \cdot \frac{2}{4} = 0.3$, and the probability that the product is even is $1 - 0.3 = 0.7$.

13. **Answer (D):** Let x denote the number of sets of quadruplets. Then $1000 = 4 \cdot x + 3 \cdot (4x) + 2 \cdot (3 \cdot 4x) = 40x$. Thus $x = 25$, and the number of babies in sets of quadruplets is $4 \cdot 25 = 100$.

14. **Answer (D):** Note that $3 < \pi < 4$, $6 < 2\pi < 7$, $9 < 3\pi < 10$, and $12 < 4\pi < 13$. Therefore there are 3 1-by-1 squares of the desired type in the strip $1 \le x \le 2$, 6 1-by-1 squares in the strip $2 \le x \le 3$, 9 1-by-1 squares in the strip $3 \le x \le 4$, and 12 1-by-1 squares in the strip $4 \le x \le 5$. Furthermore there are 2 2-by-2 squares in the strip $1 \le x \le 3$, 5 2-by-2 squares in the strip $2 \le x \le 4$, and 8 2-by-2 squares in the strip $3 \le x \le 5$. There is 1 3-by-3 square in the strip $1 \le x \le 4$, and there are 4 3-by-3 squares in the strip $2 \le x \le 5$. There are no 4-by-4 or larger squares. Thus in all there are $3 + 6 + 9 + 12 + 2 + 5 + 8 + 1 + 4 = 50$ squares of the desired type within the given region.



15. **Answer (C):** Shade the squares in a checkerboard pattern as shown in the first figure. Because consecutive numbers must be in adjacent squares, the shaded squares will contain either five odd numbers or five even numbers. Because there are only four even numbers available, the shaded squares contain the five odd numbers. Thus the sum of the numbers in all five shaded squares is $1 + 3 + 5 + 7 + 9 = 25$. Because all but the center add up to $18 = 25 - 7$, the center number must be 7. The situation described is actually possible, as the second figure demonstrates.



16. **Answer (E):** Let r be the common ratio of the geometric series; then

$$S = \frac{1}{r} + 1 + r + r^2 + \cdots = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2}.$$

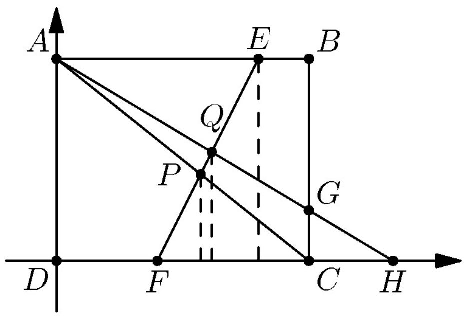
Because $S > 0$, the smallest value of S occurs when the value of $r - r^2$ is maximized. The graph of $f(r) = r - r^2$ is a downward-opening parabola with vertex $(\frac{1}{2}, \frac{1}{4})$, so the smallest possible value of S is $\frac{1}{(\frac{1}{4})} = 4$. The optimal series is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

17. **Answer (D):** Suppose that one pair of opposite faces of the cube are assigned the numbers a and b, a second pair of opposite faces are assigned the numbers c and d, and the remaining pair of opposite faces are assigned the numbers e and f. Then the needed sum of products is $ace + acf + ade + adf + bce + bcf + bde + bdf = (a+b)(c+d)(e+f)$. The sum of these three factors is $2+3+4+5+6+7 = 27$. A product of positive numbers whose sum is fixed is maximized when the factors are all equal. Thus the greatest possible value occurs when $a+b = c+d = e+f = 9$, as in $(a,b,c,d,e,f) = (2,7,3,6,4,5)$. This results in the value $9^3 = 729$.

18. **Answer (E):** A sum of consecutive integers is equal to the number of integers in the sum multiplied by their median. Note that $345 = 3 \cdot 5 \cdot 23$. If there are an odd number of integers in the sum, then the median and the number of integers must be complementary factors of 345. The only possibilities are 3 integers with median $5 \cdot 23 = 115$, 5 integers with median $3 \cdot 23 = 69$, $3 \cdot 5 = 15$ integers with median 23, and 23 integers with median $3 \cdot 5 = 15$. Having more integers in the sum would force some of the integers to be negative. If there are an even number of integers in the sum, say $2k$, then the median will be $\frac{j}{2}$,

where k and j are complementary factors of 345. The possibilities are 2 integers with median $\frac{345}{2}$, 6 integers with median $\frac{115}{2}$, and 10 integers with median $\frac{69}{2}$. Again, having more integers in the sum would force some of the integers to be negative. This gives a total of 7 solutions.

19. **Answer (D):** Triangles AEP and CFP are similar and $FP : EP = CF : AE = 3 : 4$, so $FP = \frac{3}{7}EF$. Extend $\overline{AG}$ and $\overline{FC}$ to meet at point H; then $\triangle AEQ$ and $\triangle HFQ$ are similar. Note that $\triangle HCG$ and $\triangle ABG$ are similar with sides in a ratio of $1 : 3$, so $CH = \frac{1}{3} \cdot 5$ and $FH = 3 + \frac{5}{3} = \frac{14}{3}$. Then $FQ : EQ = \frac{14}{3} : 4 = 7 : 6$, so $FQ = \frac{7}{13}FE$. Thus $PQ = FQ - FP = \left(\frac{7}{13} - \frac{3}{7}\right)FE = \frac{10}{91}FE$ and $\frac{PQ}{FE} = \frac{10}{91}$.



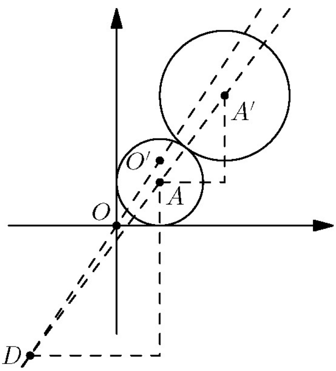
OR

Place the figure in the coordinate plane with D at the origin, A at $(0, 4)$, and C at $(5, 0)$. Then the equations of lines AC, AG, and EF are $y = -\frac{4}{5}x + 4$, $y = -\frac{3}{5}x + 4$, and $y = 2x - 4$, respectively. The intersections can be found by solving simultaneous linear equations: $P\left(\frac{20}{7}, \frac{12}{7}\right)$ and $Q\left(\frac{40}{13}, \frac{28}{13}\right)$. Because F, P, Q, and E are aligned, ratios of distances between these points are the same as ratios of the corresponding distances between their coordinates. Then

$$\frac{PQ}{FE} = \frac{\frac{40}{13} - \frac{20}{7}}{4 - 2} = \frac{10}{91}.$$

20. **Answer (C):** The scale factor for this transformation is $\frac{3}{2}$. The center of the dilation, D, must lie along ray $A'A$ (with A between A' and D), and its distance from A must be $\frac{2}{3}$ of its distance from A'. Because A is 3 units to the left of and 4 units below A', the center of the dilation must be 6 units to the left of and 8 units below A, placing it at $D(-4, -6)$. The origin is $\sqrt{(-4)^2 + (-6)^2} = 2\sqrt{13}$ units

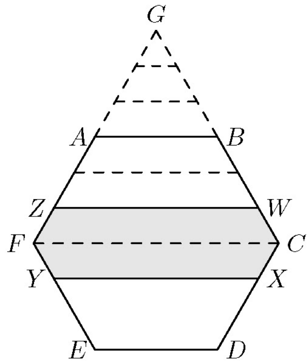
from D, so the dilation must move it half that far, or $\sqrt{13}$ units. Alternatively, note that the origin is 4 units to the right of and 6 units above D, so its image must be 6 units to the right of and 9 units above D; therefore it is located at $(2, 3)$, a distance $\sqrt{2^2 + 3^2} = \sqrt{13}$ from the origin.



21. **Answer (B):** The graph of the equation is symmetric about both axes. In the first quadrant, the equation is equivalent to $x^2 + y^2 - x - y = 0$. Completing the square gives $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$, so the graph in the first quadrant is an arc of the circle that is centered at $C(\frac{1}{2}, \frac{1}{2})$ and contains the points $A(1, 0)$ and $B(0, 1)$. Because C is the midpoint of $\overline{AB}$, the arc is a semicircle. The region enclosed by the graph in the first quadrant is the union of isosceles right triangle AOB, where $O(0, 0)$ is the origin, and a semicircle with diameter $\overline{AB}$. The triangle and the semicircle have areas $\frac{1}{2}$ and $\frac{1}{2} \cdot \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$, respectively, so the area of the region enclosed by the graph in all quadrants is $4(\frac{1}{2} + \frac{\pi}{4}) = \pi + 2$.

22. **Answer (A):** There must have been $10 + 10 + 1 = 21$ teams, and therefore there were $\binom{21}{3} = \frac{21 \cdot 20 \cdot 19}{6} = 1330$ subsets $\{A, B, C\}$ of three teams. If such a subset does not satisfy the stated condition, then it consists of a team that beat both of the others. To count such subsets, note that there are 21 choices for the winning team and $\binom{10}{2} = 45$ choices for the other two teams in the subset. This gives $21 \cdot 45 = 945$ such subsets. The required answer is $1330 - 945 = 385$. To see that such a scenario is possible, arrange the teams in a circle, and let each team beat the 10 teams that follow it in clockwise order around the circle.

23. **Answer (C):** Extend sides $\overline{CB}$ and $\overline{FA}$ to meet at G. Note that $FC = 2AB$ and $ZW = \frac{5}{3}AB$. Then the areas of $\triangle BAG$, $\triangle WZG$, and $\triangle CFG$ are in the ratio $1^2 : (\frac{5}{3})^2 : 2^2 = 9 : 25 : 36$. Thus $\frac{[ZWCF]}{[ABCF]} = \frac{36-25}{36-9} = \frac{11}{27}$, and by symmetry, $\frac{[WCXYFZ]}{[ABCDEF]} = \frac{11}{27}$ also.



OR

Suppose that $AB = 1$; then $FZ = \frac{1}{3}$ and $FC = 2$. Trapezoid $WCFZ$, which is the upper half of hexagon $WCXYFZ$, can be tiled by 11 equilateral triangles of side length $\frac{1}{3}$, and the lower half similarly, making 22 such triangles. Hexagon $ABCDEF$ can be tiled by 6 equilateral triangles of side length 1, and each of these can be tiled by 9 equilateral triangles of side length $\frac{1}{3}$, making a total of $6 \cdot 9 = 54$ small triangles. The required ratio is $\frac{22}{54} = \frac{11}{27}$.

24. **Answer (D):** Let k be the common difference for the arithmetic sequence. If $b = c$ or $c = d$, then $k = bc - ab = cd - bc$ must be a multiple of 10, so $b = c = d$. However, the two-digit integers bc and cd are then equal, a contradiction. Therefore either (b, c, d) or $(b, c, d+10)$ is an increasing arithmetic sequence.

Case 1: (b, c, d) is an increasing arithmetic sequence. In this case the additions of k to ab and bc do not involve any carries, so (a, b, c) also forms an increasing arithmetic sequence, as does (a, b, c, d). Let $n = b - a$. If $n = 1$, the possible values of a are 1, 2, 3, 4, 5, and 6. If $n = 2$, the possible values of a are 1, 2, and 3. There are no possibilities with $n \geq 3$. Thus in this case there are 9 integers that have the required property: 1234, 2345, 3456, 4567, 5678, 6789, 1357, 2468, and 3579.

Case 2: $(b, c, d + 10)$ is an increasing arithmetic sequence. In this case the addition of k to bc involves a carry, so $(a, b, c - 1)$ forms a nondecreasing arithmetic sequence, as does $(b, c-1, (d+10)-2) = (b, c-1, d+8)$. Hence $(a, b, c - 1, d + 8)$ is a nondecreasing arithmetic sequence. Again letting $n = b - a$, note that $0 \leq c = d + (9 - n) \leq 9$ and $1 \leq a = d + (8 - 3n) \leq 9$. The only integers with the required properties are 8890 with $n = 0$; 5680 and 6791 with $n = 1$; 2470, 3581, and 4692 with $n = 2$; and 1482 and 2593 with $n = 3$. Thus in this case there are 8 integers that have the required property.

The total number of integers with the required property is $9 + 8 = 17$.

25. **Answer (A):** Note that for any x, $f(x+1) = \sum_{k=2}^{10}(\lfloor kx + k\rfloor - k\lfloor x+1\rfloor) = \sum_{k=2}^{10}(\lfloor kx\rfloor + k - k\lfloor x\rfloor - k) = f(x)$. This implies that $f(x)$ is periodic with period 1. Thus the number of distinct values that $f(x)$ assumes is the same as the number of distinct values that $f(x)$ assumes for $0 \le x < 1$. For these x, $\lfloor x\rfloor = 0$, so $f(x) = \sum_{k=2}^{10}\lfloor kx\rfloor$, which is a nondecreasing function of x. This function increases at exactly those values of x expressible as a fraction of positive integers with denominator between 2 and 10. There are 31 such values between 0 and 1. They are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$. Thus $f(0) = 0$ and $f(x)$ increases 31 times for x between 0 and 1, showing that $f(x)$ assumes 32 distinct values.

Problems and solutions were contributed by Sam Baethge, Tom Butts, Barb Currier, Marta Eso, Silvia Fernandez, Chuck Garner, Jerry Grossman, Michael Khoury, Hugh Montgomery, Harold Reiter, and David Wells.

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