MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

19th Annual

AMC 10A

American Mathematics Competition 10A

Wednesday, February 7, 2018

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (B):** Computing inside to outside yields:

$$\begin{aligned}\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1 &= \left(\left(\frac{4}{3}\right)^{-1}+1\right)^{-1}+1\\ &= \left(\frac{7}{4}\right)^{-1}+1\\ &= \frac{11}{7}.\end{aligned}$$

   Note: The successive denominators and numerators of numbers obtained from this pattern are the Lucas numbers.

2. **Answer (A):** Let L, J, and A be the amounts of soda that Liliane, Jacqueline, and Alice have, respectively. The given information implies that $L = 1.50J = \frac{3}{2}J$ and $A = 1.25J = \frac{5}{4}J$, and hence $J = \frac{4}{5}A$. Then

$$L = \frac{3}{2} \cdot \frac{4}{5}A = \frac{6}{5}A = 1.20A,$$

   so Liliane has 20% more soda than Alice.

3. **Answer (E):** Converting 10! seconds to days gives

$$\frac{10!}{60 \cdot 60 \cdot 24} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 120}{60 \cdot 120 \cdot 12} = \frac{9 \cdot 8 \cdot 7}{12} = 42.$$

   Because 30 days after January 1 is January 31, 42 days after January 1 is February 12.

4. **Answer (E):** There are 4 choices for the periods in which the mathematics courses can be taken: periods 1, 3, 5; periods 1, 3, 6; periods 1, 4, 6; and periods 2, 4, 6. Each choice of periods allows $3! = 6$ ways to order the 3 mathematics courses. Therefore there are $4 \cdot 6 = 24$ ways of arranging a schedule.

5. **Answer (D):** Because the statements of Alice, Bob, and Charlie are all incorrect, the actual distance d satisfies $d < 6$, $d > 5$, and $d > 4$. Hence the actual distance lies in the interval $(5, 6)$.

6. **Answer (B):** Let N be the number of votes cast. Then $0.65N$ of them were like votes, and $0.35N$ of them were dislike votes. The current score for Sangho's video is then $0.65N - 0.35N = 0.3N = 90$. Thus $N = 90 \div (0.3) = 300$.

7. **Answer (E):** Because $4000 = 2^5 \cdot 5^3$,

$$4000 \cdot \left(\frac{2}{5}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

This product will be an integer if and only if both of the factors 2^{5+n} and 5^{3-n} are integers, which happens if and only if both exponents are nonnegative. Therefore the given expression is an integer if and only if $5+n \geq 0$ and $3-n \geq 0$. The solutions are exactly the integers satisfying $-5 \leq n \leq 3$. There are $3 - (-5) + 1 = 9$ such values.

8. **Answer (C):** Let n be the number of 5-cent coins Joe has, and let x be the requested value—the number of 25-cent coins Joe has minus the number of 5-cent coins he has. Then Joe has $(n+3)$ 10-cent coins and $(n+x)$ 25-cent coins. The given information leads to the equations

$$\begin{aligned} n + (n+3) + (n+x) &= 23 \\ 5n + 10(n+3) + 25(n+x) &= 320. \end{aligned}$$

These equations simplify to $3n + x = 20$ and $8n + 5x = 58$. Solving these equations simultaneously yields $n = 6$ and $x = 2$. Joe has 2 more 25-cent coins than 5-cent coins. Indeed, Joe has 6 5-cent coins, 9 10-cent coins, and 8 25-cent coins.

9. **Answer (E):** The length of the base $\overline{DE}$ of $\triangle ADE$ is 4 times the length of the base of a small triangle, so the area of $\triangle ADE$ is $4^2 \cdot 1 = 16$. Therefore the area of $DBCE$ is the area of $\triangle ABC$ minus the area of $\triangle ADE$, which is $40 - 16 = 24$.

10. **Answer (A):** Let

$$a = \sqrt{49 - x^2} - \sqrt{25 - x^2} \quad \text{and} \quad b = \sqrt{49 - x^2} + \sqrt{25 - x^2}.$$

Then $ab = (49 - x^2) - (25 - x^2) = 24$, so $b = \frac{24}{a} = \frac{24}{3} = 8$.

OR

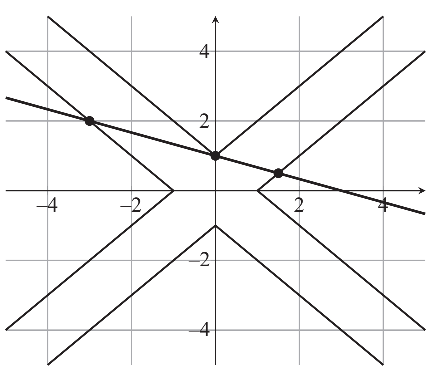
The given equation can be solved directly. Adding $\sqrt{25-x^2}$ to both sides of the equation and squaring leads to $15 = 6\sqrt{25-x^2}$. Solving for x^2 gives $x^2 = \frac{75}{4}$. Substituting this value into $\sqrt{49-x^2}+\sqrt{25-x^2}$ gives the value 8.

11. **Answer (E):** The only ways to achieve a sum of 10 by adding 7 unordered integers between 1 and 6 inclusive are (i) six 1s and one 4; (ii) five 1s, one 2, and one 3; or (iii) four 1s and three 2s. The number of ways to order the outcomes among the 7 dice are 7 in case (i), $7 \cdot 6 = 42$ in case (ii), and $\binom{7}{3} = 35$ in case (iii). There are 6^7 possible outcomes. Therefore $n = 7 + 42 + 35 = 84$.

OR

The number of ways to achieve a sum of 10 by adding 7 ordered integers between 1 and 6, inclusive, is the same as the number of ways to insert 6 bars in the spaces between stars in a row of 10 stars (with no more than one bar per space). For example, the sum $1+1+2+1+3+1+1$ corresponds to $*|*|**|*|***|*|*$. The number of ways of inserting 6 bars in the 9 spaces in a row of 10 stars is $\binom{9}{6} = 84$. (This approach is referred to as "stars and bars".)

12. **Answer (C):** The graph of the system is shown below.



The graph of the first equation is a line with x-intercept $(3, 0)$ and y-intercept $(0, 1)$. To draw the graph of the second equation, consider the equation quadrant by quadrant. In the first quadrant $x > 0$ and $y > 0$, and thus the second equation is equivalent to $|x - y| = 1$, which in turn is equivalent to $y = x \pm 1$. Its graph consists of the rays with endpoints $(0,1)$ and $(1,0)$, as shown. In the second quadrant $x<0$ and $y>0$. The corresponding graph is the reflection of the first quadrant graph across the y-axis. The rest of the graph can be sketched by further reflections of the first-quadrant graph across the coordinate axes, resulting in the figure shown. There are 3 intersection points: $(-3,2)$, $(0,1)$, and $\left(\frac{3}{2},\frac{1}{2}\right)$, as shown.

OR

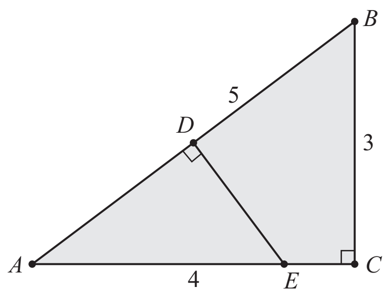
The second equation implies that $x=y\pm 1$ or $x=-y\pm 1$. There are four cases:

- If $x=y+1$, then $(y+1)+3y=3$, so $(x,y)=\left(\frac{3}{2},\frac{1}{2}\right)$.
- If $x=y-1$, then $(y-1)+3y=3$, so $(x,y)=(0,1)$.
- If $x=-y+1$, then $(-y+1)+3y=3$, so again $(x,y)=(0,1)$.
- If $x=-y-1$, then $(-y-1)+3y=3$, so $(x,y)=(-3,2)$.

It may be checked that each of these ordered pairs actually satisfies the given equations, so the total number of solutions is 3.

13. **Answer (D):** The paper's long edge $\overline{AB}$ is the hypotenuse of right triangle ACB, and the crease lies along the perpendicular bisector of $\overline{AB}$. Because $AC>BC$, the crease hits $\overline{AC}$ rather than $\overline{BC}$. Let D be the midpoint of $\overline{AB}$, and let E be the intersection of $\overline{AC}$ and the line through D perpendicular to $\overline{AB}$. Then the crease in the paper is $\overline{DE}$. Because $\triangle ADE \sim \triangle ACB$, it follows that $\frac{DE}{AD}=\frac{CB}{AC}=\frac{3}{4}$. Thus

$$DE = AD\cdot\frac{CB}{AC}=\frac{5}{2}\cdot\frac{3}{4}=\frac{15}{8}.$$



14. **Answer (A):** Because the powers-of-3 terms greatly dominate the powers-of-2 terms, the given fraction should be close to

$$\frac{3^{100}}{3^{96}}=3^4=81.$$

Note that

$$\left(3^{100}+2^{100}\right)-81\left(3^{96}+2^{96}\right)=2^{100}-81\cdot 2^{96}=(16-81)\cdot 2^{96}<0,$$

so the given fraction is less than 81. On the other hand

$$\left(3^{100}+2^{100}\right)-80\left(3^{96}+2^{96}\right)=3^{96}(81-80)-2^{96}(80-16)=3^{96}-2^{102}.$$

Because $3^2>2^3$,

$$3^{96}=\left(3^2\right)^{48}>\left(2^3\right)^{48}=2^{144}>2^{102},$$

it follows that

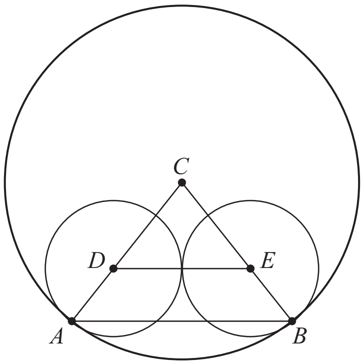
$$\left(3^{100}+2^{100}\right)-80\left(3^{96}+2^{96}\right)>0,$$

and the given fraction is greater than 80. Therefore the greatest integer less than or equal to the given fraction is 80.

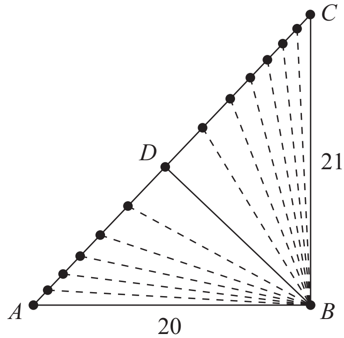
15. **Answer (D):** Let C be the center of the larger circle, and let D and E be the centers of the two smaller circles, as shown. Points C, D, and A are collinear because the radii are perpendicular to the common tangent at the point of tangency, and so are C, E, and B. These points form two isosceles triangles that share a vertex angle. Thus $\triangle CAB\sim\triangle CDE$, and therefore $\frac{AB}{DE}=\frac{CA}{CD}$, so

$$AB=\frac{DE\cdot CA}{CD}=\frac{(5+5)\cdot 13}{13-5}=\frac{65}{4},$$

and the requested sum is $65+4=69$.



16. **Answer (D):** The area of $\triangle ABC$ is 210. Let D be the foot of the altitude from B to $\overline{AC}$. By the Pythagorean Theorem, $AC = \sqrt{20^2 + 21^2} = 29$, so $210 = \frac{1}{2} \cdot 29 \cdot BD$, and $BD = 14\frac{14}{29}$. Two segments of every length from 15 through 19 can be constructed from B to $\overline{AC}$. In addition to these 10 segments and the 2 legs, there is a segment of length 20 from B to a point on $\overline{AC}$ near C, for a total of 13 segments with integer length.



17. **Answer (C):** If $1 \in S$, then S can have only 1 element, not 6 elements. If 2 is the least element of S, then 2, 3, 5, 7, 9, and 11 are available to be in S, but 3 and 9 cannot both be in S, so the largest possible size of S is 5. If 3 is the least element, then 3, 4, 5, 7, 8, 10, and 11 are available, but at most one of 4 and 8 can be in S and at most one of 5 and 10 can be in S, so again S has size at most 5. The set $S = \{4, 6, 7, 9, 10, 11\}$ has the required property, so 4 is the least possible element of S.

OR

At most one integer can be selected for S from each of the following 6 groups: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, and $\{11\}$. Because S consists of 6 distinct integers, exactly one integer must be selected from each of these 6 groups. Therefore 7, 9, and 11 must be in S. Because 9 is in S, 3 must not be in S. This implies that either 6 or 12 must be selected from the second group, so neither 1 nor 2 can be selected from the first group. If 4 is selected from the first group, the collection of integers $\{4, 5, 6, 7, 9, 11\}$ is one possibility for the set S. Therefore 4 is the least possible element of S.

Note: The two collections given in the solutions are the only ones with least element 4 that have the given property. This problem is a special case of the following result of Paul Erdős from the 1930s:

Given n integers $a_1, a_2, \ldots, a_n$, no one of them dividing any other, with $a_1 < a_2 < \cdots < a_n \le 2n$, let the integer k be determined by the inequalities $3^k < 2n < 3^{k+1}$. Then $a_1 \ge 2^k$, and this bound is sharp.

18. **Answer (D):** Let S be the set of integers, both negative and nonnegative, having the given form. Increasing the value of a_i by 1 for $0 \le i \le 7$ creates a one-to-one correspondence between S and the ternary (base 3) representation of the integers from 0 through $3^8 - 1$, so S contains $3^8 = 6561$ elements. One of those is 0, and by symmetry, half of the others are positive, so S contains $1 + \frac{1}{2} \cdot (6561 - 1) = 3281$ elements.

OR

First note that if an integer N can be written in this form, then $N - 1$ can also be written in this form as long as not all the a_i in the representation of N are equal to -1. A procedure to alter the representation of N so that it will represent $N - 1$ instead is to find the least value of i such that $a_i \ne -1$, reduce the value of that a_i by 1, and set $a_i = 1$ for all lower values of i. By the formula for the sum of a finite geometric series, the greatest integer that can be written in the given form is

$$\frac{3^8 - 1}{3 - 1} = 3280.$$

Therefore, 3281 nonnegative integers can be written in this form, namely all the integers from 0 through 3280, inclusive. (The negative integers from -3280 through -1 can also be written in this way.)

OR

Think of the indicated sum as an expansion in base 3 using "digits" -1, 0, and 1. Note that the leftmost digit a_k of any positive integer that can be written in this form cannot be negative and therefore must be 1. Then there are 3 choices for each of the remaining k digits to the right of a_k, resulting in 3^k positive integers that can be written in the indicated form. Thus there are

$$\sum_{k=0}^{7} 3^k = \frac{3^8 - 1}{3 - 1} = 3280$$

positive numbers of the indicated form. Because 0 can also be written in this form, the number of nonnegative integers that can be written in the indicated form is 3281.

19. **Answer (E):** For $m \in \{11, 13, 15, 17, 19\}$, let $p(m)$ denote the probability that m^n has units digit 1, where n is chosen at random from the set $S = \{1999, 2000, 2001, \ldots, 2018\}$. Then the desired probability is equal to $\frac{1}{5}\big(p(11) + p(13) + p(15) + p(17) + p(19)\big)$. Because any positive integral power of 11 always has units digit 1, $p(11) = 1$, and because any positive integral power of 15 always has units digit 5, $p(15) = 0$. Note that S has 20 elements, exactly 5 of which are congruent to j mod 4 for each of $j = 0, 1, 2, 3$. The units digits of powers of 13 and 17 cycle in groups of 4. More precisely,

$$\left(13^k \bmod 10\right)_{k=1999}^{2018} = (7, 1, 3, 9, 7, 1, \ldots, 3, 9)$$

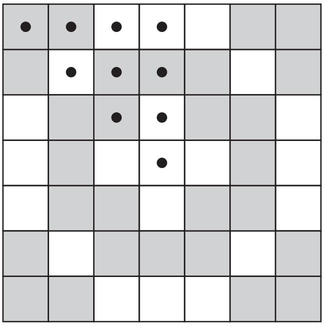
and

$$\left(17^k \bmod 10\right)_{k=1999}^{2018} = (3, 1, 7, 9, 3, 1, \ldots, 7, 9).$$

Thus $p(13) = p(17) = \frac{5}{20} = \frac{1}{4}$. Finally, note that the units digit of 19^k is 1 or 9, according to whether k is even or odd, respectively. Thus $p(19) = \frac{1}{2}$. Hence the requested probability is

$$\frac{1}{5}\left(1 + \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{2}\right) = \frac{2}{5}.$$

20. **Answer (B):** None of the squares that are marked with dots in the sample scanning code shown below can be mapped to any other marked square by reflections or non-identity rotations. Therefore these 10 squares can be arbitrarily colored black or white in a symmetric scanning code, with the exception of "all black" and "all white". On the other hand, reflections or rotations will map these squares to all the other squares in the scanning code, so once these 10 colors are specified, the symmetric scanning code is completely determined. Thus there are $2^{10} - 2 = 1022$ symmetric scanning codes.



OR

The diagram below shows the orbits of each square under rotations and reflections. Because the scanning code must look the same under these transformations, all squares in the same orbit must get the same color, but one is free to choose the color for each orbit, except for the choice of "all black" and "all white". Because there are 10 orbits, there are $2^{10} - 2 = 1022$ symmetric scanning codes.

A	B	C	D	C	B	A
B	E	F	G	F	E	B
C	F	H	I	H	F	C
D	G	I	J	I	G	D
C	F	H	I	H	F	C
B	E	F	G	F	E	B
A	B	C	D	C	B	A

21. **Answer (E):** Solving the second equation for x^2 gives $x^2 = y + a$, and substituting into the first equation gives $y^2 + y + (a - a^2) = 0$. The polynomial in y can be factored as $(y + (1 - a))(y + a)$, so the solutions are $y = a - 1$ and $y = -a$. (Alternatively, the solutions can be obtained using the quadratic formula.) The corresponding equations for x are $x^2 = 2a - 1$ and $x^2 = 0$. The second equation always has the solution $x = 0$, corresponding to the point of tangency at the vertex of the parabola $y = x^2 - a$. The first equation has 2 solutions if and only if $a > \frac{1}{2}$, corresponding to the 2 symmetric intersection points of the parabola with the circle. Thus the two curves intersect at 3 points if and only if $a > \frac{1}{2}$.

OR

Substituting the value for y from the second equation into the first equation yields

$$x^2 + \left(x^2 - a\right)^2 = a^2,$$

which is equivalent to

$$x^2\left(x^2 - (2a - 1)\right) = 0.$$

The first factor gives the solution $x = 0$, and the second factor gives 2 other solutions if $a > \frac{1}{2}$ and no other solutions if $a \le \frac{1}{2}$. Thus there are 3 solutions if and only if $a > \frac{1}{2}$.

22. **Answer (D):** Because $\gcd(a, b) = 24 = 2^3 \cdot 3$ and $\gcd(b, c) = 36 = 2^2 \cdot 3^2$, it follows that a is divisible by 2 and 3 but not by 3^2. Similarly, because $\gcd(b, c) = 2^2 \cdot 3^2$ and $\gcd(c, d) = 54 = 2 \cdot 3^3$, it follows that d is divisible by 2 and 3 but not by 2^2. Therefore $\gcd(d, a) = 2 \cdot 3 \cdot n$, where n is a product of primes that do not include 2 or 3. Because $70 < \gcd(d, a) < 100$ and n is an integer, it must be that $12 \le n \le 16$, so $n = 13$, and 13 must also be a divisor of a. The conditions are satisfied if $a = 2^3 \cdot 3 \cdot 13 = 312$, $b = 2^3 \cdot 3^2 = 72$, $c = 2^2 \cdot 3^3 = 108$, and $d = 2 \cdot 3^3 \cdot 13 = 702$.

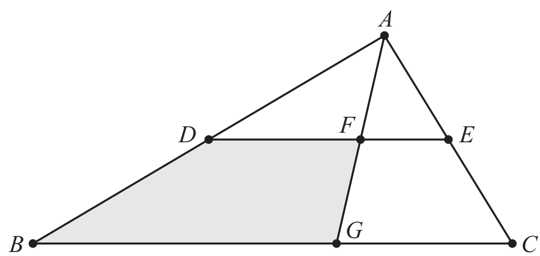
23. **Answer (D):** Let the triangle's vertices in the coordinate plane be $(4, 0)$, $(0, 3)$, and $(0, 0)$, with $[0, s] \times [0, s]$ representing the unplanted portion of the field. The equation of the hypotenuse is $3x+4y-12 = 0$, so the distance from (s, s), the corner of S closest to the hypotenuse, to this line is given by
$$\frac{|3s + 4s - 12|}{\sqrt{3^2 + 4^2}}.$$
Setting this equal to 2 and solving for s gives $s = \frac{22}{7}$ and $s = \frac{2}{7}$, and the former is rejected because the square must lie within the triangle. The unplanted area is thus $\left(\frac{2}{7}\right)^2 = \frac{4}{49}$, and the requested fraction is
$$1 - \frac{\frac{4}{49}}{\frac{1}{2} \cdot 4 \cdot 3} = \frac{145}{147}.$$

OR

Let the given triangle be described as $\triangle ABC$ with the right angle at B and $AB = 3$. Let D be the vertex of the square that is in the interior of the triangle, and let s be the edge length of the square. Then two sides of the square along with line segments $\overline{AD}$ and $\overline{CD}$ decompose $\triangle ABC$ into four regions. These regions are a triangle with base 5 and height 2, the unplanted square with side s, a right triangle with legs s and $3 - s$, and a right triangle with legs s and $4 - s$. The sum of the areas of these four regions is
$$\frac{1}{2} \cdot 5 \cdot 2 + s^2 + \frac{1}{2}s(3 - s) + \frac{1}{2}s(4 - s) = 5 + \frac{7}{2}s,$$

and the area of $\triangle ABC$ is 6. Solving $5 + \frac{7}{2}s = 6$ for s gives $s = \frac{2}{7}$, and the solution concludes as above.

24. **Answer (D):** Because AB is $\frac{5}{6}$ of $AB + AC$, it follows from the Angle Bisector Theorem that DF is $\frac{5}{6}$ of DE, and BG is $\frac{5}{6}$ of BC. Because trapezoids $FDBG$ and $EDBC$ have the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$. Furthermore, the area of $\triangle ADE$ is $\frac{1}{4}$ of the area of $\triangle ABC$, so its area is 30, and the area of trapezoid $EDBC$ is $120 - 30 = 90$. Therefore the area of quadrilateral $FDBG$ is $\frac{5}{6} \cdot 90 = 75$.



Note: The figure (not drawn to scale) shows the situation in which $\angle ACB$ is acute. In this case $BC \approx 59.0$ and $\angle BAC \approx 151^\circ$. It is also possible for $\angle ACB$ to be obtuse, with $BC \approx 41.5$ and $\angle BAC \approx 29^\circ$. These values can be calculated using the Law of Cosines and the sine formula for area.

25. **Answer (D):** The equation $C_n - B_n = A_n^2$ is equivalent to

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \left(\frac{10^n - 1}{9} \right)^2 .$$

Dividing by $10^n - 1$ and clearing fractions yields

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

As this must hold for two different values n_1 and n_2, there are two such equations, and subtracting them gives

$$\left(9c - a^2\right)\left(10^{n_1} - 10^{n_2}\right) = 0.$$

The second factor is non-zero, so $9c - a^2 = 0$ and thus $9b - 9c - a^2 = 0$. From this it follows that $c = \left(\frac{a}{3}\right)^2$ and $b = 2c$. Hence digit a must be 3, 6, or 9, with corresponding values 1, 4, or 9 for c, and 2, 8, or

18 for b. The case $b = 18$ is invalid, so there are just two triples of possible values for a, b, and c, namely $(3, 2, 1)$ and $(6, 8, 4)$. In fact, in these cases, $C_n - B_n = A_n^2$ for *all* positive integers n; for example, $4444 - 88 = 4356 = 66^2$. The second triple has the greater coordinate sum, $6 + 8 + 4 = 18$.

Problems and solutions were contributed by David Altizio, Risto Atanasov, George Bauer, Ivan Borsenco, Thomas Butts, Barbara Currier, Steve Dunbar, Marta Eso, Zachary Franco, Peter Gilchrist, Jerrold Grossman, Chris Jeuell, Jonathan Kane, Joe Kennedy, Michael Khoury, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, Michael Tang, David Wells, and Carl Yerger.

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