1

2D GEOMETERY

2001

11. **(C)** The n^{th} ring can be partitioned into four rectangles: two containing $2n+1$ unit squares and two containing $2n-1$ unit squares. So there are

$$2(2n+1)+2(2n-1)=8n$$

unit squares in the n^{th} ring. Thus, the 100^{th} ring has $8\cdot 100=800$ unit squares.

OR

The n^{th} ring can be obtained by removing a square of side $2n-1$ from a square of side $2n+1$. So it contains

$$(2n+1)^2-(2n-1)^2=(4n^2+4n+1)-(4n^2-4n+1)=8n$$

unit squares.

2016A

9. **Answer (D):** There are

$$1+2+\cdots+N=\frac{N(N+1)}{2}$$

coins in the array. Therefore $N(N+1)=2\cdot 2016=4032$. Because $N(N+1)\approx N^2$, it follows that $N\approx\sqrt{4032}\approx\sqrt{2^{12}}=2^6=64$. Indeed, $63\cdot 64=4032$, so $N=63$ and the sum of the digits of N is 9.

2000

12. **Answer (C):** Calculating the number of squares in the first few figures uncovers a pattern. Figure 0 has $2(0) + 1 = 2(0^2) + 1$ squares, figure 1 has $2(1) + 3 = 2(1^2) + 3$ squares, figure 2 has $2(1 + 3) + 5 = 2(2^2) + 5$ squares, and figure 3 has $2(1 + 3 + 5) + 7 = 2(3^2) + 7$ squares. In general, the number of unit squares in figure n is

$$2(1 + 3 + 5 + \cdots + (2n - 1)) + 2n + 1 = 2(n^2) + 2n + 1.$$

Therefore, the figure 100 has $2(100^2) + 2 \cdot 100 + 1 = 20201$.

OR

Each figure can be considered as a large square with identical small pieces deleted from each of the four corners. Figure 1 has $3^2 - 4(1)$ unit squares, figure 2 has $5^2 - 4(1 + 2)$ unit squares, and figure 3 has $7^2 - 4(1 + 2 + 3)$ unit squares. In general, figure n has

$$(2n - 1)^2 - 4(1 + 2 + \cdots + n) = (2n + 1)^2 - 2n(n + 1) \text{ unit squares.}$$

Thus figure 100 has $201^2 - 200(101) = 20201$ unit squares.

OR

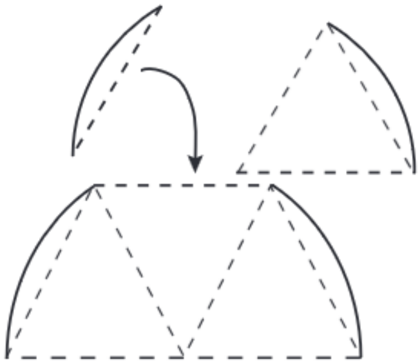
The number of unit squares in figure n is the sum of the first n positive odd integers plus the sum of the first $n + 1$ positive odd integers. Since the sum of the first k positive odd integers is k^2, figure n has $n^2 + (n + 1)^2$ unit squares. So figure 100 has $100^2 + 101^2 = 20201$ unit squares.

2004B

12. **(A)** The area of the annulus is the difference between the areas of the two circles, which is $\pi b^2 - \pi c^2$. Because the tangent $\overline{XZ}$ is perpendicular to the radius $\overline{OZ}$, $b^2 - c^2 = a^2$, so the area is πa^2.

2005A 12. **(B)** The trefoil is constructed of four equilateral triangles and four circular segments, as shown. These can be combined to form four $60°$ circular sectors. Since the radius of the circle is 1, the area of the trefoil is

$$\frac{4}{6}\left(\pi \cdot 1^2\right) = \frac{2}{3}\pi.$$



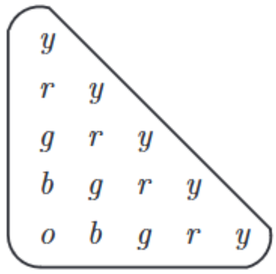
2009A 12. **Answer (C):** Let x be the length of $\overline{BD}$. By the triangle inequality on $\triangle BCD$, $5 + x > 17$, so $x > 12$. By the triangle inequality on $\triangle ABD$, $5 + 9 > x$, so $x < 14$. Since x must be an integer, $x = 13$.

2013A 12. **Answer (C):** Because $\overline{EF}$ is parallel to $\overline{AB}$, it follows that $\triangle FEC$ is similar to $\triangle ABC$ and $FE = FC$. Thus half of the perimeter of $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

2014A 12. **Answer (C):** Each of the 6 sectors has radius 3 and central angle $120°$. Their combined area is $6 \cdot \frac{1}{3} \cdot \pi \cdot 3^2 = 18\pi$. The hexagon can be partitioned into 6 equilateral triangles each having side length 6, so the hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 6^2 = 54\sqrt{3}$. The shaded region has area $54\sqrt{3} - 18\pi$.

2000

13. **Answer (B):** To avoid having two yellow pegs in the same row or column, there must be exactly one yellow peg in each row and in each column. Hence, starting at the top of the array, the peg in the first row must be yellow, the second peg of the second row must be yellow, the third peg of the third row must be yellow, etc. To avoid having two red pegs in some row, there must be a red peg in each of rows 2,3,4, and 5. The red pegs must be in the first position of the second row, the second position of the third row, etc. Continuation yields exactly one ordering that meets the requirements, as shown.



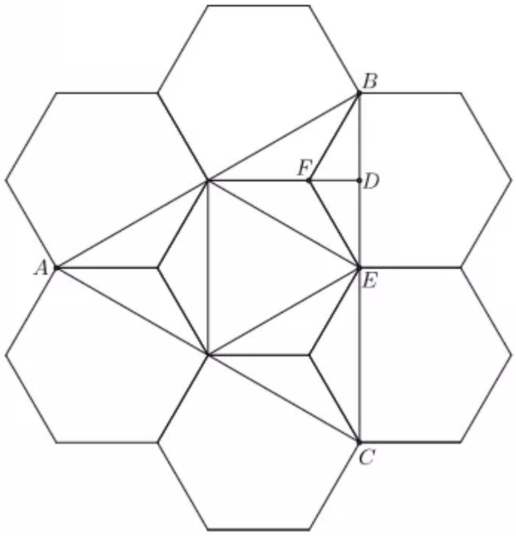
2009B

13. **Answer (C):** Define a rotation of the pentagon to be a sequence that starts with $\overline{AB}$ on the x-axis and ends when $\overline{AB}$ is on the x-axis the first time thereafter. Because the pentagon has perimeter 23 and $2009 = 23 \cdot 87 + 8$, it follows that after 87 rotations, point A will be at $x = 23 \cdot 87 = 2001$ and point B will be at $x = 2001 + 3 = 2004$. Points C and D will next touch the x-axis at $x = 2004 + 4 = 2008$ and $x = 2008 + 6 = 2014$, respectively. Therefore a point on $\overline{CD}$ will touch $x = 2009$.

2014A

13. **Answer (C):** The three squares each have area 1, and $\triangle ABC$ has area $\frac{\sqrt{3}}{4}$. Note that $\angle EAF = 360° - 60° - 2 \cdot 90° = 120°$. Thus the altitude from A in isosceles $\triangle EAF$ partitions the triangle into two $30-60-90°$ right triangles, each with hypotenuse 1. It follows that $\triangle EAF$ has base $EF = \sqrt{3}$ and altitude $\frac{1}{2}$, so its area is $\frac{\sqrt{3}}{4}$. Similarly, triangles GCH and DBI each have area $\frac{\sqrt{3}}{4}$. Therefore the area of hexagon $DEFGHI$ is $3 \cdot \frac{\sqrt{3}}{4} + 3 \cdot 1 + \frac{\sqrt{3}}{4} = 3 + \sqrt{3}$.

2014B 13. **Answer (B):** Label points E and F as shown in the figure, and let D be the midpoint of $\overline{BE}$. Because $\triangle BFD$ is a $30-60-90^\circ$ triangle with hypotenuse 1, the length of $\overline{BD}$ is $\frac{\sqrt{3}}{2}$, and therefore $BC = 2\sqrt{3}$. It follows that the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4} \cdot (2\sqrt{3})^2 = 3\sqrt{3}$.



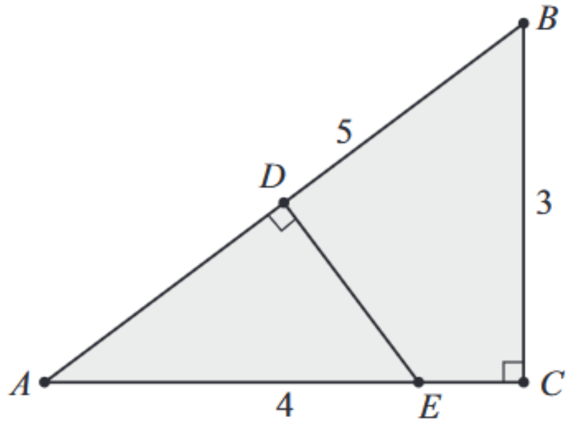
OR

Notice that $AE = 3$ since AE is composed of a hexagon side (length 1) and the longest diagonal of a hexagon (length 2). Triangle ABE is $30-60-90^\circ$, so $BE = \frac{3}{\sqrt{3}} = \sqrt{3}$. The area of $\triangle ABC$ is $AE \cdot BE = 3\sqrt{3}$.

2018A

13. **Answer (D):** The paper's long edge $\overline{AB}$ is the hypotenuse of right triangle ACB, and the crease lies along the perpendicular bisector of $\overline{AB}$. Because $AC > BC$, the crease hits $\overline{AC}$ rather than $\overline{BC}$. Let D be the midpoint of $\overline{AB}$, and let E be the intersection of $\overline{AC}$ and the line through D perpendicular to $\overline{AB}$. Then the crease in the paper is $\overline{DE}$. Because $\triangle ADE \sim \triangle ACB$, it follows that $\frac{DE}{AD} = \frac{CB}{AC} = \frac{3}{4}$. Thus

$$DE = AD \cdot \frac{CB}{AC} = \frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.$$



2005B

14. **(C)** Drop $\overline{MQ}$ perpendicular to $\overline{BC}$. Then $\triangle MQC$ is a 30–60–90° triangle, so $MQ = \sqrt{3}/2$, and the area of $\triangle CDM$ is

$$\frac{1}{2}\left(2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

OR

Triangles ABC and CDM have equal bases. Because M is the midpoint of $\overline{AC}$, the ratio of the altitudes from M and from A is $1/2$. So the area of $\triangle CDM$ is half of the area of $\triangle ABC$. Since

$$\text{Area}(\triangle ABC) = \frac{\sqrt{3}}{4} \cdot 2^2 = \sqrt{3}, \quad \text{we have} \quad \text{Area}(\triangle CDM) = \frac{\sqrt{3}}{2}.$$

2006A

14. **(B)** The top of the largest ring is 20 cm above its bottom. That point is 2 cm below the top of the next ring, so it is 17 cm above the bottom of the next ring. The additional distances to the bottoms of the remaining rings are 16 cm, 15 cm, . . . , 1 cm. Thus the total distance is

$$20 + (17 + 16 + \cdots + 2 + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 17 \cdot 9 = 173\,\text{cm}.$$

OR

The required distance is the sum of the outside diameters of the 18 rings minus a 2-cm overlap for each of the 17 pairs of consecutive rings. This equals

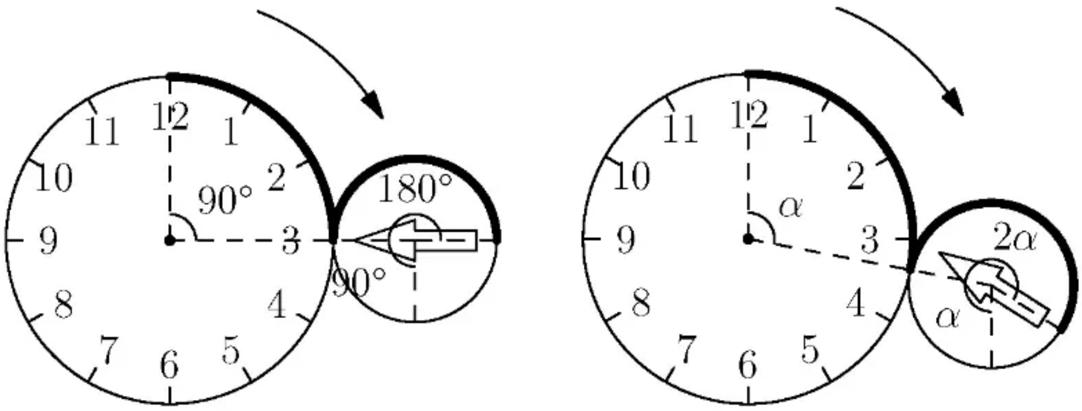
$$(3+4+5+\cdots+20)-2\cdot17 = (1+2+3+4+5+\cdots+20)-3-34 = \frac{20 \cdot 21}{2}-37 = 173\,\text{cm}.$$

2009A

14. **Answer (A):** Let the lengths of the shorter and longer side of each rectangle be x and y, respectively. The outer and inner squares have side lengths $y + x$ and $y - x$, respectively, and the ratio of their side lengths is $\sqrt{4} = 2$. Therefore $y + x = 2(y - x)$, from which $y = 3x$.

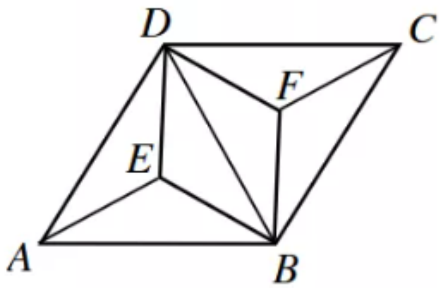
2015A

14. **Answer (C):** The circumference of the disk is half the circumference of the clock face. As the disk rolls $\frac{1}{4}$ of the way around the circumference of the clock face (from 12 o'clock to 3 o'clock), the disk rolls through $\frac{1}{2}$ of its own circumference. At that point, the arrow of the disk is pointing at the point of tangency, so the arrow on the disk will have turned $\frac{3}{4}$ of one revolution. In general, as the disk rolls through an angle α around the clock face, the arrow on the disk turns through an angle 3α on the disk. The arrow will again be pointing in the upward vertical direction when the disk has turned through 1 complete revolution, and the angle traversed on the clock face is $\frac{1}{3}$ of the way around the face. The point of tangency will be at 4 o'clock.



2006B

15. **(C)** Since $\angle BAD = 60°$, isosceles $\triangle BAD$ is also equilateral. As a consequence, $\triangle AEB$, $\triangle AED$, $\triangle BED$, $\triangle BFD$, $\triangle BFC$, and $\triangle CFD$ are congruent. These six triangles have equal areas and their union forms rhombus $ABCD$, so each has area $24/6 = 4$. Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, so its area is 8.

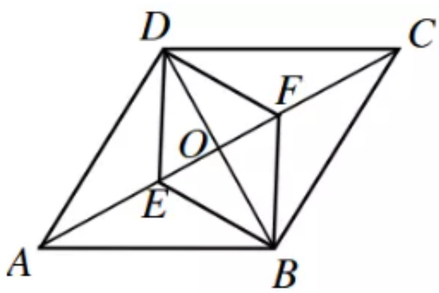


OR

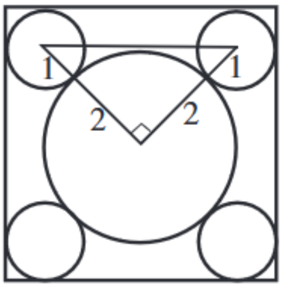
Let the diagonals of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus intersect at right angles, $\triangle ABO$ is a $30-60-90°$ triangle. Therefore $AO = \sqrt{3} \cdot BO$. Because AO and BO are half the length of the longer diagonals of rhombi $ABCD$ and $BFDE$, respectively, it follows that

$$\frac{\text{Area}(BFDE)}{\text{Area}(ABCD)} = \left(\frac{BO}{AO}\right)^2 = \frac{1}{3}.$$

Thus the area of rhombus $BFDE$ is $(1/3)(24) = 8$.



2007A 15. **Answer (B):** Let s be the length of a side of the square. Consider an isosceles right triangle with vertices at the centers of the circle of radius 2 and two of the circles of radius 1. This triangle has legs of length 3, so its hypotenuse has length $3\sqrt{2}$.



The length of a side of the square is 2 more than the length of this hypotenuse, so $s = 2 + 3\sqrt{2}$. Hence the area of the square is

$$s^2 = (2 + 3\sqrt{2})^2 = 22 + 12\sqrt{2}.$$

OR

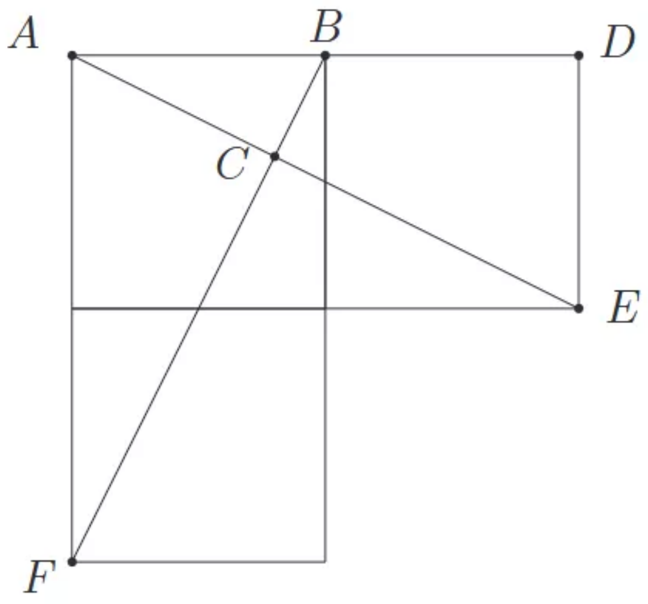
The distance from a vertex of the square to the center of the nearest small circle is $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the distance between the centers of two small circles in opposite corners of the square is $1 + 4 + 1 = 6$. Therefore each diagonal of the square has length $6 + 2\sqrt{2}$, and each side has length

$$s = \frac{6 + 2\sqrt{2}}{\sqrt{2}} = 2 + 3\sqrt{2}.$$

The area of the square is consequently $s^2 = \left(2 + 3\sqrt{2}\right)^2 = 22 + 12\sqrt{2}$.

2012A

15. **Answer (B):** Place the figure on the coordinate plane with A at the origin, B on the positive x-axis, and label the other points as shown. Then the equation of line AE is $y = -\frac{1}{2}x$, and the equation of line BF is $y = 2x - 2$. Solving the simultaneous equations shows that $C = (\frac{4}{5}, -\frac{2}{5})$. Therefore $\triangle ABC$ has base $AB = 1$ and altitude $\frac{2}{5}$, so its area is $\frac{1}{5}$.



OR

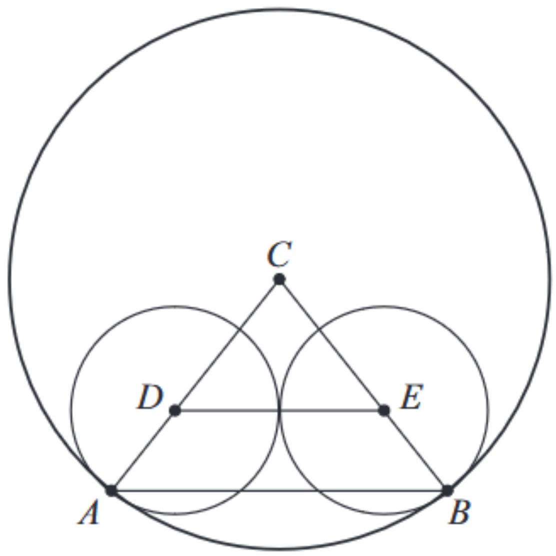
Congruent right triangles AED and FBA have the property that their corresponding legs are perpendicular; hence their hypotenuses are perpendicular. Therefore $\angle ACB$ is a right angle and $\triangle ACB$ is similar to $\triangle FAB$. Because $AB = 1$ and $BF = \sqrt{5}$, the ratio of the area of $\triangle ACB$ to that of $\triangle FAB$ is 1 to 5. The area of $\triangle FAB$ is 1, so the area of $\triangle ACB$ is $\frac{1}{5}$.

2018A

15. **Answer (D):** Let C be the center of the larger circle, and let D and E be the centers of the two smaller circles, as shown. Points C, D, and A are collinear because the radii are perpendicular to the common tangent at the point of tangency, and so are C, E, and B. These points form two isosceles triangles that share a vertex angle. Thus $\triangle CAB \sim \triangle CDE$, and therefore $\frac{AB}{DE} = \frac{CA}{CD}$, so

$$AB = \frac{DE \cdot CA}{CD} = \frac{(5+5) \cdot 13}{13-5} = \frac{65}{4},$$

and the requested sum is $65 + 4 = 69$.



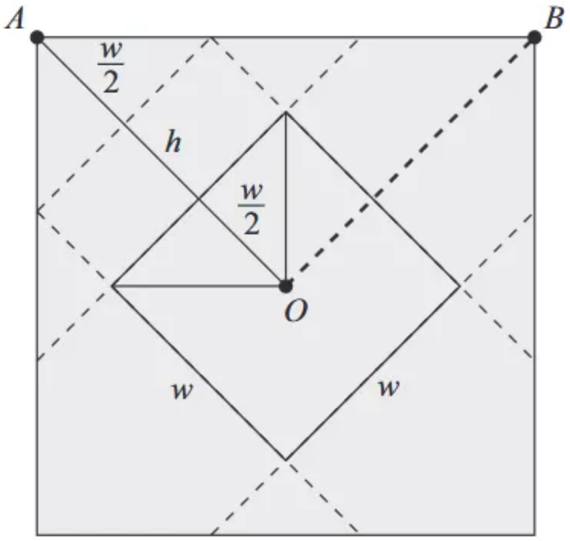
2018B

15. **Answer (A):** The figure shows that the distance AO from a corner of the wrapping paper to the center is

$$\frac{w}{2} + h + \frac{w}{2} = w + h.$$

The side of the wrapping paper, $\overline{AB}$ in the figure, is the hypotenuse of a $45-45-90°$ right triangle, so its length is $\sqrt{2} \cdot AO = \sqrt{2}(w+h)$. Therefore the area of the wrapping paper is

$$\left(\sqrt{2}(w+h)\right)^2 = 2(w+h)^2.$$



OR

The area of the wrapping paper, excluding the four small triangles indicated by the dashed lines, is equal to the surface area of the box, which is $2w^2 + 4wh$. The four triangles are isosceles right triangles with leg length h, so their combined area is $4 \cdot \frac{1}{2}h^2 = 2h^2$. Thus the total area of the wrapping paper is $2w^2 + 4wh + 2h^2 = 2(w+h)^2$.

2014B

15. **Answer (A):** Let $AD = \sqrt{3}$. Because $\angle ADE = 30°$, it follows that $AE = 1$ and $DE = 2$. Now $\angle EDF = 30°$ and $\angle DEF = 120°$, so $\triangle DEF$ is isosceles and $EF = 2$. Thus the area of $\triangle DEF$ (with $\overline{EF}$ viewed as the base) is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$, and the desired ratio is $\frac{\sqrt{3}}{\sqrt{3} \cdot 2\sqrt{3}} = \frac{\sqrt{3}}{6}$.

2016A 13. **Answer (B):** The total number of seats moved to the right among the five friends must equal the total number of seats moved to the left. One of Dee and Edie moved some number of seats to the right, and the other moved the same number of seats to the left. Because Bea moved two seats to the right and Ceci moved one seat to the left, Ada must also move one seat to the left upon her return. Because her new seat is an end seat and its number cannot be 5, it must be seat 1. Therefore Ada occupied seat 2 before she got up. The order before moving was Bea-Ada-Ceci-Dee-Edie (or Bea-Ada-Ceci-Edie-Dee), and the order after moving was Ada-Ceci-Bea-Edie-Dee (or Ada-Ceci-Bea-Dee-Edie).