10

PROBABILITY

2015B

11. **Answer (B):** There are four one-digit primes (2, 3, 5, and 7), which can be used to form $4^2 = 16$ two-digit numbers with prime digits. Of these two-digit numbers, only 23, 37, 53, and 73 are prime. So there are $4 + 16 = 20$ numbers less than 100 whose digits are prime, and $4 + 4 = 8$ of them are prime. The probability is $\frac{8}{20} = \frac{2}{5}$.

2004B 11. **(C)** There are $8 \cdot 8 = 64$ ordered pairs that can represent the top numbers on the two dice. Let m and n represent the top numbers on the dice. Then $mn > m+n$ implies that $mn - m - n > 0$, that is,

$$1 < mn - m - n + 1 = (m-1)(n-1).$$

This inequality is satisfied except when $m = 1$, $n = 1$, or when $m = n = 2$. There are 16 ordered pairs (m, n) excluded by these conditions, so the probability that the product is greater than the sum is

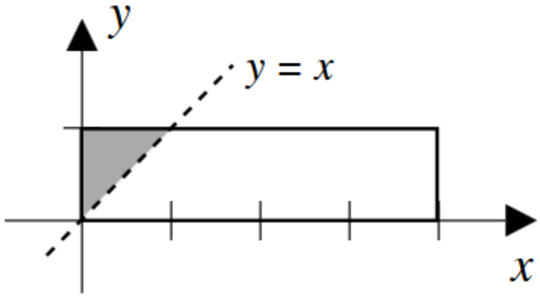
$$\frac{64-16}{64} = \frac{48}{64} = \frac{3}{4}.$$

2018A 11. **Answer (E):** The only ways to achieve a sum of 10 by adding 7 unordered integers between 1 and 6 inclusive are (i) six 1s and one 4; (ii) five 1s, one 2, and one 3; or (iii) four 1s and three 2s. The number of ways to order the outcomes among the 7 dice are 7 in case (i), $7 \cdot 6 = 42$ in case (ii), and $\binom{7}{3} = 35$ in case (iii). There are 6^7 possible outcomes. Therefore $n = 7 + 42 + 35 = 84$.

OR

The number of ways to achieve a sum of 10 by adding 7 ordered integers between 1 and 6, inclusive, is the same as the number of ways to insert 6 bars in the spaces between stars in a row of 10 stars (with no more than one bar per space). For example, the sum $1+1+2+1+3+1+1$ corresponds to $*|*|**|*|***|*|*$. The number of ways of inserting 6 bars in the 9 spaces in a row of 10 stars is $\binom{9}{6} = 84$. (This approach is referred to as "stars and bars".)

2003A 12. (A) The point (x, y) satisfies $x < y$ if and only if it belongs to the shaded triangle bounded by the lines $x = y$, $y = 1$, and $x = 0$, the area of which is $1/2$. The ratio of the area of the triangle to the area of the rectangle is $\frac{1/2}{4} = \frac{1}{8}$.



2005B 12. **(E)** Exactly one die must have a prime face on top, and the other eleven must have 1's. The prime die can be any one of the twelve, and the prime can be 2, 3, or 5. Thus the probability of a prime face on any one die is $1/2$, and the probability of a prime product is

$$12\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^{11} = \left(\frac{1}{6}\right)^{10}.$$

2013B 12. **Answer (B):** The five sides of the pentagon are congruent, and the five congruent diagonals are longer than the sides. Once one segment is selected, 4 of the 9 remaining segments have the same length as that segment. Therefore the requested probability is $\frac{4}{9}$.

2016A 10. **Answer (B):** Let the inner rectangle's length be x feet; then its area is x square feet. The middle region has area $3(x+2) - x = 2x + 6$, so the difference in the arithmetic sequence is equal to $(2x+6) - x = x+6$. The outer region has area $5(x+4) - 3(x+2) = 2x + 14$, so the difference in the arithmetic sequence is also equal to $(2x+14) - (2x+6) = 8$. From $x + 6 = 8$, it follows that $x = 2$. The regions have areas 2, 10, and 18.

2016B 12. **Answer (D):** The product of two integers is odd if and only if both integers are odd. Thus the probability that the product is odd is $\frac{3}{5} \cdot \frac{2}{4} = 0.3$, and the probability that the product is even is $1 - 0.3 = 0.7$.

2006A 13. **(D)** Let x represent the amount the player wins if the game is fair. The chance of an even number is $1/2$, and the chance of matching this number on the second roll is $1/6$. So the probability of winning is $(1/2)(1/6) = 1/12$. Therefore $(1/12)x = \$5$ and $x = \$60$.

2011B 13. **Answer (D):** Consider all ordered pairs (a, b) with each of the numbers a and b in the closed interval $[-20, 10]$. These pairs fill a 30×30 square in the coordinate plane, with an area of 900 square units. Ordered pairs in the first and third quadrants have the desired property, namely $a \cdot b > 0$. The areas of the portions of the 30×30 square in the first and third quadrants are $10^2 = 100$ and $20^2 = 400$, respectively. Therefore the probability of a positive product is $\frac{100+400}{900} = \frac{5}{9}$.

OR

Each of the numbers is positive with probability $\frac{1}{3}$ and negative with probability $\frac{2}{3}$. Their product is positive if and only if both numbers are positive or both are negative, so the requested probability is $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$.

2017B 14. **Answer (D):** An integer will have a remainder of 1 when divided by 5 if and only if the units digit is either 1 or 6. The randomly selected positive integer will itself have a units digit of each of the numbers

from 0 through 9 with equal probability. This digit of N alone will determine the units digit of N^{16}. Computing the 16th power of each of these 10 digits by squaring the units digit four times yields one 0, one 5, four 1s, and four 6s. The probability is therefore $\frac{8}{10} = \frac{4}{5}$.

Note: This result also follows from Fermat's Little Theorem.

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Note: This result also follows from Fermat's Little Theorem.

2003A 15. (C) Of the $\frac{100}{2} = 50$ integers that are divisible by 2, there are $\lfloor \frac{100}{6} \rfloor = 16$ that are divisible by both 2 and 3. So there are $50 - 16 = 34$ that are divisible by 2 and not by 3, and $34/100 = 17/50$.

2005B 15. **(D)** There are

$$\binom{8}{2} = \frac{8!}{6! \cdot 2!} = 28$$

ways to choose the bills. A sum of at least \$20 is obtained by choosing both \$20 bills, one of the \$20 bills and one of the six smaller bills, or both \$10 bills. Hence the probability is

$$\frac{1 + 2 \cdot 6 + 1}{28} = \frac{14}{28} = \frac{1}{2}.$$

2017A

15. **Answer (C):** Half of the time Laurent will pick a number between 2017 and 4034, in which case the probability that his number will be greater than Chloé's number is 1. The other half of the time, he will pick a number between 0 and 2017, and by symmetry his number will be the larger one in half of those cases. Therefore the requested probability is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

OR

The choices of numbers can be represented in the coordinate plane by points in the rectangle with vertices at $(0, 0)$, $(2017, 0)$, $(2017, 4034)$, and $(0, 4034)$. The portion of the rectangle representing the event that Laurent's number is greater than Chloé's number is the portion above the line segment with endpoints $(0, 0)$ and $(2017, 2017)$. This area is $\frac{3}{4}$ of the area of the entire rectangle, so the requested probability is $\frac{3}{4}$.