15

ARITHMETIC

2006B 11. **(C)** Since $n!$ contains the product $2 \cdot 5 \cdot 10 = 100$ whenever $n \geq 10$, it suffices to determine the tens digit of

$$7! + 8! + 9! = 7!(1 + 8 + 8 \cdot 9) = 5040(1 + 8 + 72) = 5040 \cdot 81.$$

This is the same as the units digit of $4 \cdot 1$, which is 4.

2003B 13. (E) Let $y = \clubsuit(x)$. Since $x \leq 99$, we have $y \leq 18$. Thus if $\clubsuit(y) = 3$, then $y = 3$ or $y = 12$. The 3 values of x for which $\clubsuit(x) = 3$ are 12, 21, and 30, and the 7 values of x for which $\clubsuit(x) = 12$ are 39, 48, 57, 66, 75, 84, and 93. There are 10 values in all.

2002B 14. **(B)** We have

$$N = \sqrt{(5^2)^{64} \cdot (2^6)^{25}} = 5^{64} \cdot 2^{3 \cdot 25} = (5 \cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048 = 2048\underbrace{000 \cdots 0}_{64 \text{ digits}}.$$

The zeros do not contribute to the sum, so the sum of the digits of N is $2+4+8 = 14$.

2013B

14. **Answer (E):** The equation $x\clubsuit y = y\clubsuit x$ is equivalent to $x^2y - xy^2 = y^2x - yx^2$. This equation is euivalent to gives $2xy(x-y) = 0$. This equation will hold exactly if $x = 0$, $y = 0$, or $x = y$. The solution set consists of three lines: the x-axis, the y-axis, and the line $x = y$.

2018A

14. **Answer (A):** Because the powers-of-3 terms greatly dominate the powers-of-2 terms, the given fraction should be close to

$$\frac{3^{100}}{3^{96}} = 3^4 = 81.$$

Note that

$$\left(3^{100} + 2^{100}\right) - 81\left(3^{96} + 2^{96}\right) = 2^{100} - 81 \cdot 2^{96} = (16 - 81) \cdot 2^{96} < 0,$$

so the given fraction is less than 81. On the other hand

$$\left(3^{100} + 2^{100}\right) - 80\left(3^{96} + 2^{96}\right) = 3^{96}(81-80) - 2^{96}(80-16) = 3^{96} - 2^{102}.$$

Because $3^2 > 2^3$,

$$3^{96} = \left(3^2\right)^{48} > \left(2^3\right)^{48} = 2^{144} > 2^{102},$$

it follows that

$$\left(3^{100} + 2^{100}\right) - 80\left(3^{96} + 2^{96}\right) > 0,$$

and the given fraction is greater than 80. Therefore the greatest integer less than or equal to the given fraction is 80.

2005A

15. **(E)** Written as a product of primes, we have

$$3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

A cube that is a factor has a prime factorization of the form $2^p \cdot 3^q \cdot 5^r \cdot 7^s$, where p, q, r, and s are all multiples of 3. There are 3 possible values for p, which are 0, 3, and 6. There are 2 possible values for q, which are 0 and 3. The only value for r and for s is 0. Hence there are $6 = 3 \cdot 2 \cdot 1 \cdot 1$ distinct cubes that divide $3! \cdot 5! \cdot 7!$. They are

$$1 = 2^0 3^0 5^0 7^0, \quad 8 = 2^3 3^0 5^0 7^0, \quad 27 = 2^0 3^3 5^0 7^0,$$
$$64 = 2^6 3^0 5^0 7^0, \quad 216 = 2^3 3^3 5^0 7^0, \quad \text{and} \quad 1728 = 2^6 3^3 5^0 7^0.$$

2011B

15. **Answer (E):** If $x \neq 0$, then I is false:

$$x \,@\, (y+z) = \frac{x+(y+z)}{2} \neq \frac{x+y+x+z}{2} = \frac{x+y}{2} + \frac{x+z}{2} = (x \,@\, y) + (x \,@\, z).$$

On the other hand, II and III are true for all values of x, y and z:

$$x + (y \,@\, z) = x + \frac{y+z}{2} = \frac{2x+y+z}{2} = \frac{(x+y)+(x+z)}{2} = (x+y) \,@\, (x+z),$$

and

$$x \,@\, (y \,@\, z) = \frac{x + \frac{y+z}{2}}{2} = \frac{\left(\frac{2x+y+z}{2}\right)}{2} = \frac{\frac{x+y}{2} + \frac{x+z}{2}}{2} = (x \,@\, y) \,@\, (x \,@\, z)$$