17

ALGEBRA WORD PROBLEMS

2000

11. **Answer (C):** There are five prime numbers between 4 and 18: 5,7,11,13, and 17. Hence the product of any two of these is odd and the sum is even. Because $xy-(x+y)=(x-1)(y-1)-1$ increases as either x or y increases (since both x and y are bigger than 1), the answer must be an odd number that is no smaller than $23=5\cdot 7-(5+7)$ and no larger than $191=13\cdot 17-(13+17)$. The only possibility among the options is 119, and indeed $119=11\cdot 13-(11+13)$.

2001

12. **(D)** In any triple of consecutive integers, at least one is even and one is a multiple of 3. Therefore, the product of the three integers is both even and a multiple of 3. Since 7 is adivisor of the product, the numbers 6, 14, 21, and 42 must also be divisors of the product. However, 28 contains two factors of 2, and n need not. For example, $5\cdot 6\cdot 7$ is divisible by 7, but not by 28.

2002B

11. **(B)** Let $n-1$, n, and $n+1$ denote the three integers. Then

$$(n-1)n(n+1)=8(3n).$$

Since $n\neq 0$, we have $n^2-1=24$. It follows that $n^2=25$ and $n=5$. Thus,

$$(n-1)^2+n^2+(n+1)^2=16+25+36=77.$$

2009B

11. **Answer (A):** Because the digit 5 appears three times, 5 must be the middle digit of any such palindrome. In the first three digits each of 2, 3, and 5 must appear once and the order in which they appear determines the last three digits. Since there are $3! = 6$ ways to order three distinct digits the number of palindromes is 6.

2012A

12. **Answer (A):** There were $200 \cdot 365 = 73000$ non-leap days in the 200-year time period from February 7, 1812 to February 7, 2012. One fourth of those years contained a leap day, except for 1900, so there were $\frac{1}{4} \cdot 200 - 1 = 49$ leap days during that time. Therefore Dickens was born 73049 days before a Tuesday. Because the same day of the week occurs every 7 days and $73049 = 7 \cdot 10435 + 4$, the day of Dickens' birth (February 7, 1812) was 4 days before a Tuesday, which was a Friday.

2003A

13. (A) Let a, b, and c be the three numbers. Replace a by four times the sum of the other two to get

$$4(b+c)+b+c=20, \quad \text{so} \quad b+c=4.$$

Then replace b with $7c$ to get

$$7c+c=4, \quad \text{so} \quad c=\frac{1}{2}.$$

The other two numbers are $b = 7/2$ and $a = 16$, and the product of the three is $16 \cdot 7/2 \cdot 1/2 = 28$.

OR

Let the first, second, and third numbers be x, $7x$, and $32x$, respectively. Then $40x = 20$ so $x = \frac{1}{2}$ and the product is

$$(32)(7)x^3 = (32)(7)\left(\frac{1}{8}\right) = 28.$$

2005B

13. **(C)** Between 1 and 2005, there are 668 multiples of 3, 501 multiples of 4, and 167 multiples of 12. So there are

$$(668 - 167) + (501 - 167) = 835$$

numbers between 1 and 2005 that are integer multiples of 3 or of 4 but not of 12.

2013A

13. **Answer (B):** Each such three-digit number must have the form aba, where a and b are digits and $a \neq 0$. Such a number will not be divisible by 5 if and only if $a \neq 5$. If a is equal to 1, 2, 3, or 4, then any of the ten choices for b satisfies the requirement. If a is equal to 6, 7, 8, or 9, then there are 8, 6, 4, or 2 choices for b, respectively. This results in $4 \cdot 10 + 8 + 6 + 4 + 2 = 60$ numbers.

2001

13. **(E)** The last four digits (GHIJ) are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A + B + C = 9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The three digits in DEF are 864, 642, or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such telephone number is 810-642-9753.

2004B

13. **(B)** The height in millimeters of any stack with an odd number of coins has a 5 in the hundredth place. The height of any two coins has an odd digit in the tenth place and a zero in the hundredth place. Therefore any stack with zeros in both its tenth and hundredth places must consist of a number of coins that is a multiple of 4. The highest stack of 4 coins has a height of $4(1.95) = 7.8$ mm, and the shortest stack of 12 coins has a height of $12(1.35) = 16.2$ mm, so no number other than 8 can work. Note that a stack of 8 quarters has a height of $8(1.75) = 14$ mm.

2016B

13. **Answer (D):** Let x denote the number of sets of quadruplets. Then $1000 = 4 \cdot x + 3 \cdot (4x) + 2 \cdot (3 \cdot 4x) = 40x$. Thus $x = 25$, and the number of babies in sets of quadruplets is $4 \cdot 25 = 100$.

2003A 14. (A) The largest single-digit primes are 5 and 7, but neither 75 nor 57 is prime. Using 3, 7, and 73 gives 1533, whose digits have a sum of 12.

2014B 14. **Answer (D):** Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c - a)$, it is a multiple of 9. Therefore m is a multiple of 495. Because m is at most a 3-digit number and a is not equal to 0, $m = 495$. Therefore $c - a = 5$. Because $a + b + c \leq 7$, the only possible abc is 106, so $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$.

OR

Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c - a)$, $c - a$ is a multiple of 5. Because $a \geq 1$ and $a+b+c \leq 7$, it follows that $c = 6$ and $a = 1$. Therefore $b = 0$, so $a^2+b^2+c^2 = 37$.

2002B 15. **(E)** The numbers $A - B$ and $A + B$ are both odd or both even. However, they are also both prime, so they must both be odd. Therefore, one of A and B is odd and the other even. Because A is a prime between $A - B$ and $A + B$, A must be the odd prime. Therefore, $B = 2$, the only even prime. So $A - 2$, A, and $A + 2$ are consecutive odd primes and thus must be 3, 5, and 7. The sum of the four primes 2, 3, 5, and 7 is the prime number 17.

2010B 15. **Answer (C):** If Jesse answered R questions correctly and W questions incorrectly, then $R + W \leq 50$, and Jesse's score is $99 = 4R - W \geq 4R - (50 - R) = 5R - 50$. Thus $5R \leq 149$, and because R is an integer, $R \leq 29$, Jesse could achieve a score of 99 by answering 29 questions correctly and 17 incorrectly, leaving 4 answers blank.

2015A

15. **Answer (B):** Because $\frac{x+1}{y+1} = \frac{11}{10} \cdot \frac{x}{y}$, it follows that $10y - 11x - xy = 0$ and so $(10 - x)(11 + y) = 110 = 2 \cdot 5 \cdot 11$. The only possible values of $10 - x$ are 5, 2, and 1 because x and y are positive integers. Thus the possible values of x are 5, 8, and 9. Of the resulting fractions $\frac{5}{11}$, $\frac{8}{44}$, and $\frac{9}{99}$, only the first is in simplest terms.

2015B

15. **Answer (B):** Let h be the number of horses and c be the number of cows. There are then $3h$ people, $9h$ ducks, and $4c$ sheep in Hamlet. The total population of Hamlet is $13h + 5c$, where h and c are whole numbers. A number N can be the population only if there exists a whole number value for h such that $N - 13h$ is a whole number multiple of 5. This is possible for all the provided numbers except 47, as follows: $41 - 13 \cdot 2 = 5 \cdot 3$, $59 - 13 \cdot 3 = 5 \cdot 4$, $61 - 13 \cdot 2 = 5 \cdot 7$, and $66 - 13 \cdot 2 = 5 \cdot 8$

None of 47, $47 - 13 = 34$, $47 - 13 \cdot 2 = 21$, and $47 - 13 \cdot 3 = 8$ is a multiple of 5. Therefore 47 cannot be the population of Hamlet.

Note: In fact, 47 is the largest number that cannot be the population.

2016B 15. **Answer (C):** Shade the squares in a checkerboard pattern as shown in the first figure. Because consecutive numbers must be in adjacent squares, the shaded squares will contain either five odd numbers or five even numbers. Because there are only four even numbers available, the shaded squares contain the five odd numbers. Thus the sum of the numbers in all five shaded squares is $1 + 3 + 5 + 7 + 9 = 25$. Because all but the center add up to $18 = 25 - 7$, the center number must be 7. The situation described is actually possible, as the second figure demonstrates.