3

2D WORD PROBLEMS

2003B

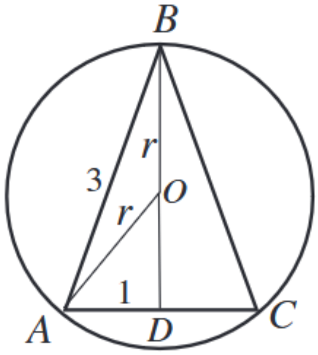
11. (A) The two lines have equations

$$y - 15 = 3(x - 10) \quad \text{and} \quad y - 15 = 5(x - 10).$$

The x-intercepts, obtained by setting $y = 0$ in the respective equations, are 5 and 7. The distance between the points $(5, 0)$ and $(7, 0)$ is 2.

2007B

11. **Answer (C):** Let $\overline{BD}$ be an altitude of the isosceles $\triangle ABC$, and let O denote the center of the circle with radius r that passes through A, B, and C, as shown.



Then

$$BD = \sqrt{3^2 - 1^2} = 2\sqrt{2} \quad \text{and} \quad OD = 2\sqrt{2} - r.$$

Since $\triangle ADO$ is a right triangle, we have

$$r^2 = 1^2 + \left(2\sqrt{2} - r\right)^2 = 1 + 8 - 4\sqrt{2}r + r^2, \quad \text{and} \quad r = \frac{9}{4\sqrt{2}} = \frac{9}{8}\sqrt{2}.$$

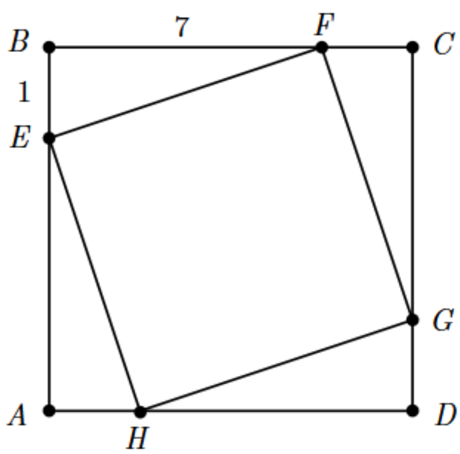
As a consequence, the circle has area

$$\left(\frac{9}{8}\sqrt{2}\right)^2 \pi = \frac{81}{32}\pi.$$

2011A

11. **Answer (B):** Without loss of generality, assume that F lies on $\overline{BC}$ and that $EB = 1$. Then $AE = 7$ and $AB = 8$. Because $EFGH$ is a square, $BF = AE = 7$, so the hypotenuse $\overline{EF}$ of $\triangle EBF$ has length $\sqrt{1^2 + 7^2} = \sqrt{50}$. The ratio of the area of $EFGH$ to that of $ABCD$ is therefore

$$\frac{EF^2}{AB^2} = \frac{50}{64} = \frac{25}{32}.$$



2012A

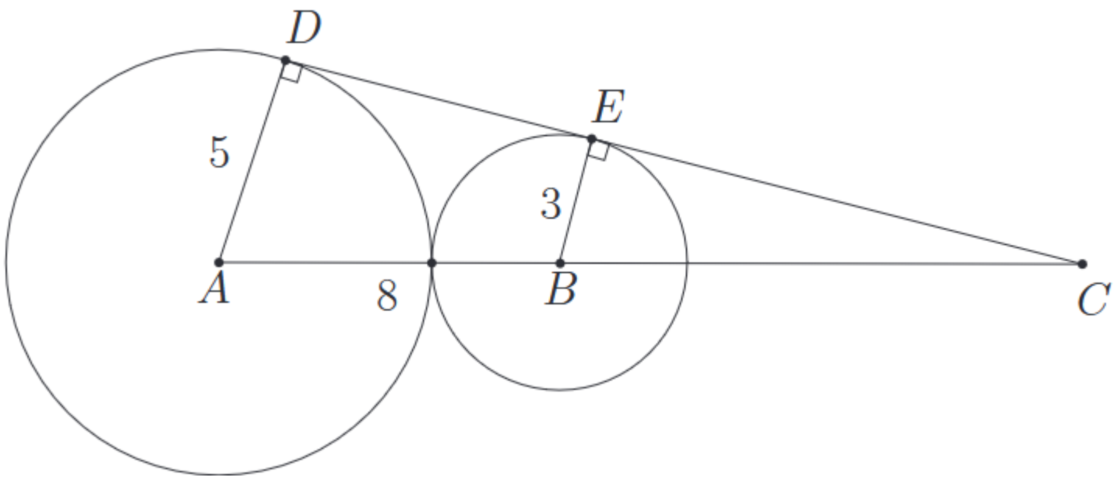
11. **Answer (D):** Let D and E be the points of tangency to circles A and B, respectively, of the common tangent line that intersects ray AB at point C. Then $AD = 5$, $BE = 3$, and $AB = 5 + 3 = 8$. Because right triangles ADC and BEC are similar, it follows that

$$\frac{BC}{AC} = \frac{BE}{AD},$$

so

$$\frac{BC}{BC + 8} = \frac{3}{5}.$$

Solving gives $BC = 12$.



2016B

11. **Answer (B):** Let x be the number of posts along the shorter side; then there are $2x$ posts along the longer side. When counting the number of posts on all the sides of the garden, each corner post is counted twice, so $2x + 2(2x) = 20 + 4$. Solving this equation gives $x = 4$. Thus the dimensions of the rectangle are $(4 - 1) \cdot 4 = 12$ yards by $(8 - 1) \cdot 4 = 28$ yards. The requested area is given by the product of these dimensions, $12 \cdot 28 = 336$ square yards.

2006B

12. **(E)** Substituting $x = 1$ and $y = 2$ into the equations gives

$$1 = \frac{2}{4} + a \quad \text{and} \quad 2 = \frac{1}{4} + b.$$

It follows that

$$a + b = \left(1 - \frac{2}{4}\right) + \left(2 - \frac{1}{4}\right) = 3 - \frac{3}{4} = \frac{9}{4}.$$

OR

Because

$$a = x - \frac{y}{4} \quad \text{and} \quad b = y - \frac{x}{4} \quad \text{we have} \quad a + b = \frac{3}{4}(x + y).$$

Since $x = 1$ when $y = 2$, this implies that $a + b = \frac{3}{4}(1 + 2) = \frac{9}{4}$.

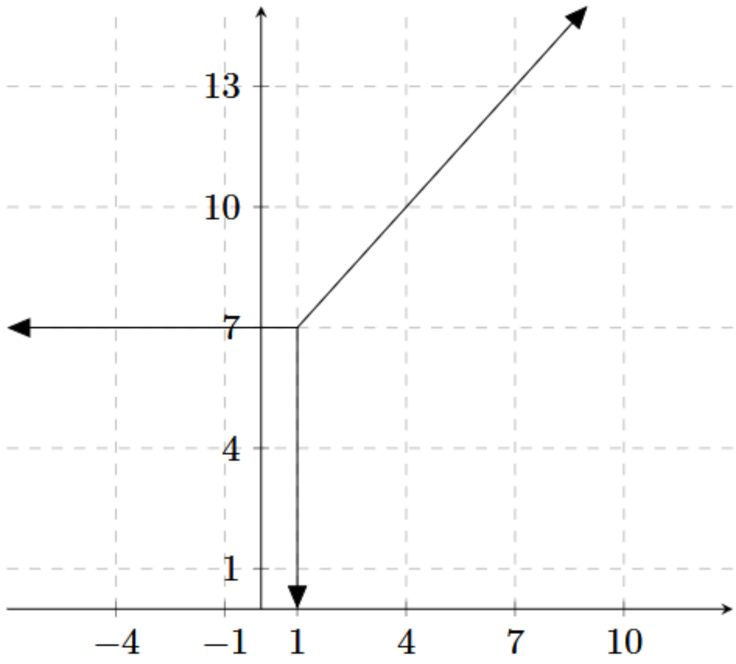
2012B

12. **Answer (A):** There were $200 \cdot 365 = 73000$ non-leap days in the 200-year time period from February 7, 1812 to February 7, 2012. One fourth of those years contained a leap day, except for 1900, so there were $\frac{1}{4} \cdot 200 - 1 = 49$ leap days during that time. Therefore Dickens was born 73049 days before a Tuesday. Because the same day of the week occurs every 7 days and $73049 = 7 \cdot 10435 + 4$, the day of Dickens' birth (February 7, 1812) was 4 days before a Tuesday, which was a Friday.

2015B

12. **Answer (A):** The circle intersects the line $y = -x$ at the points $A = (-5, 5)$ and $B = (5, -5)$. Segment $\overline{AB}$ is a chord of the circle and contains 11 points with integer coordinates.

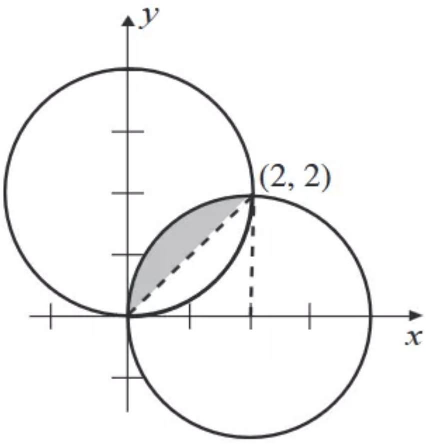
2017A 12. **Answer (E):** Suppose that the two larger quantities are the first and the second. Then $3 = x + 2 \geq y - 4$. This is equivalent to $x = 1$ and $y \leq 7$, and its graph is the downward-pointing ray with endpoint $(1, 7)$. Similarly, if the two larger quantities are the first and third, then $3 = y - 4 \geq x + 2$. This is equivalent to $y = 7$ and $x \leq 1$, and its graph is the leftward-pointing ray with endpoint $(1, 7)$. Finally, if the two larger quantities are the second and third, then $x + 2 = y - 4 \geq 3$. This is equivalent to $y = x + 6$ and $x \geq 1$, and its graph is the ray with endpoint $(1, 7)$ that points upward and to the right. Thus the graph consists of three rays with common endpoint $(1, 7)$.



Note: This problem is related to a relatively new area of mathematics called tropical geometry.

2007B

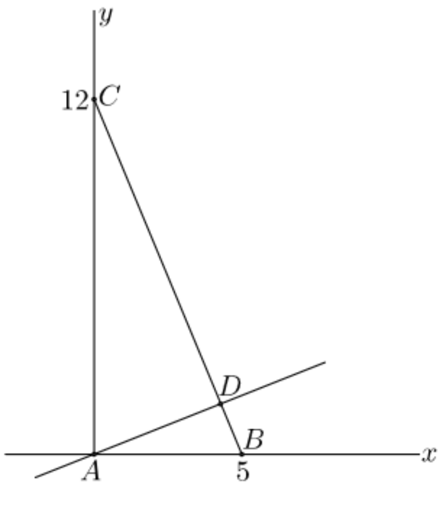
13. **Answer (D):** The two circles intersect at $(0,0)$ and $(2,2)$, as shown.



Half of the region described is formed by removing an isosceles right triangle of leg length 2 from a quarter of one of the circles. Because the quarter-circle has area $(1/4)\pi(2)^2 = \pi$ and the triangle has area $(1/2)(2)^2 = 2$, the area of the region is $2(\pi - 2)$.

2015B

13. **Answer (E):** Label the vertices of the triangle $A = (0,0)$, $B = (5,0)$, and $C = (0,12)$. By the Pythagorean Theorem $BC = 13$. Two altitudes are 5 and 12. Let $\overline{AD}$ be the third altitude. The area of this triangle is 30, so $\frac{1}{2} \cdot AD \cdot BC = 30$. Therefore $AD = \frac{2 \cdot 30}{BC} = \frac{60}{13}$. The sum of the lengths of the altitudes is $5 + 12 + \frac{60}{13} = \frac{281}{13}$.



2008B

14. **Answer (B):** Because $\triangle OAB$ is a $30-60-90^\circ$ triangle, we have $BA = \frac{5\sqrt{3}}{3}$. Let A' and B' be the images of A and B, respectively, under the rotation. Then

$B' = (0,5)$, $\overline{B'A'}$ is horizontal, and $B'A' = BA = \frac{5\sqrt{3}}{3}$. Hence A' is in the second quadrant and

$$A' = \left(-\frac{5}{3}\sqrt{3}, 5\right).$$

2011A

14. **Answer (B):** Let d be the sum of the numbers rolled. The conditions are satisfied if and only if $\pi\left(\frac{d}{2}\right)^2 < \pi d$, that is, $d < 4$. Of the 36 equally likely outcomes for the roll of the two dice, one has a sum of 2 and two have sums of 3. Thus the desired probability is $\frac{1+2}{36} = \frac{1}{12}$.

2011B

14. **Answer (C):** Let x and y be the length and width of the parking lot, respectively. Then $xy = 168$ and $x^2 + y^2 = 25^2$. Note that

$$(x+y)^2 = x^2 + y^2 + 2xy = 25^2 + 2 \cdot 168 = 961.$$

Hence the perimeter is $2(x+y) = 2 \cdot \sqrt{961} = 62$.

Note that the dimensions of the parking lot are 7 and 24 meters.

2012A

14. **Answer (B):** Separate the modified checkerboard into two parts: the first 30 columns and the last column. The larger section consists of rows, each containing 15 black squares. The last column contains 16 black squares. Thus the total number of black squares is $31 \cdot 15 + 16 = 481$.

OR

There are 16 rows that have 16 black squares and 15 rows that have 15 black squares, so the total number of black squares is $16^2 + 15^2 = 481$.

2012B 14. **Answer (B):** Separate the modified checkerboard into two parts: the first 30 columns and the last column. The larger section consists of rows, each containing 15 black squares. The last column contains 16 black squares. Thus the total number of black squares is $31 \cdot 15 + 16 = 481$.

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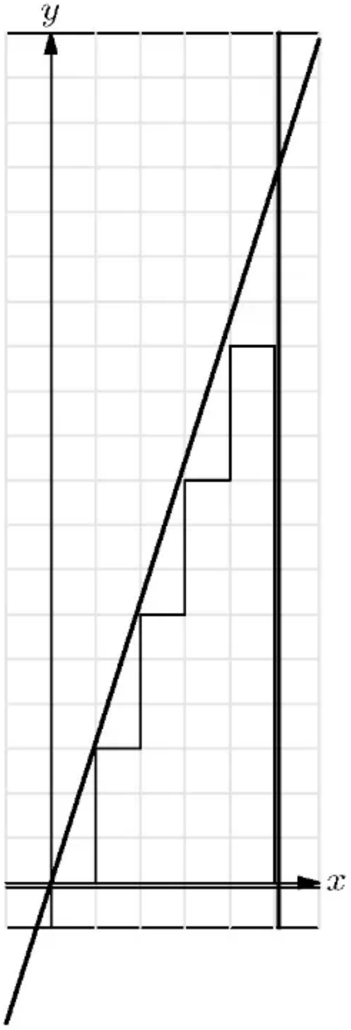
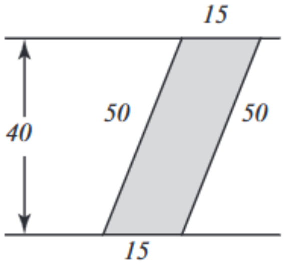
2016B 14. **Answer (D):** Note that $3 < \pi < 4$, $6 < 2\pi < 7$, $9 < 3\pi < 10$, and $12 < 4\pi < 13$. Therefore there are 3 1-by-1 squares of the desired type in the strip $1 \le x \le 2$, 6 1-by-1 squares in the strip $2 \le x \le 3$, 9 1-by-1 squares in the strip $3 \le x \le 4$, and 12 1-by-1 squares in the strip $4 \le x \le 5$. Furthermore there are 2 2-by-2 squares in the strip $1 \le x \le 3$, 5 2-by-2 squares in the strip $2 \le x \le 4$, and 8 2-by-2 squares in the strip $3 \le x \le 5$. There is 1 3-by-3 square in the strip $1 \le x \le 4$, and there are 4 3-by-3 squares in the strip $2 \le x \le 5$. There are no 4-by-4 or larger squares. Thus in all there are $3 + 6 + 9 + 12 + 2 + 5 + 8 + 1 + 4 = 50$ squares of the desired type within the given region.



2001

15. **(C)** The crosswalk is in the shape of a parallelogram with base 15 feet and altitude 40 feet, so its area is $15 \times 40 = 600\ \text{ft}^2$. But viewed another way, the parallelogram has base 50 feet and altitude equal to the distance between the stripes, so this distance must be $600/50 = 12$ feet.



2013B

15. **Answer (B):** Let s be the side length of the triangle and h the side length of the hexagon. The hexagon can be subdivided into 6 equilateral triangles by drawing segments from the center of the hexagon to each vertex. Because the areas of the large triangle and hexagon are equal, the triangles in the hexagon each have area $\frac{1}{6}$ of the area of the large triangle. Thus

$$\frac{h}{s} = \sqrt{\frac{1}{6}} \quad \text{so} \quad h = \frac{\sqrt{6}}{6}s.$$

The perimeter of the triangle is $a = 3s$ and the perimeter of the hexagon is $b = 6h = \sqrt{6}s$, so

$$\frac{a}{b} = \frac{3s}{\sqrt{6}\,s} = \frac{\sqrt{6}}{2}.$$

2017B

15. **Answer (E):** Triangles ADE and ABE have the same area because they share the base $\overline{AE}$ and, by symmetry, they have the same height. By the Pythagorean Theorem, $AC = 5$. Because $\triangle ABE \sim \triangle ACB$, the ratio of their areas is the square of the ratio of their corresponding sides. Their hypotenuses have lengths 3 and 5, respectively, so their areas are in the ratio 9 to 25. The area of $\triangle ACB$ is half that of the rectangle, so the area of $\triangle ABE$ is $\frac{9}{25} \cdot 6 = \frac{54}{25}$. Thus the area of $\triangle ADE$ is also $\frac{54}{25}$.