8

COMBINATIONS

2013A

11. **Answer (A):** Let n be the number of student council members. Because there are 10 ways of choosing the two-person welcoming committee, it follows that $10 = \binom{n}{2} = \frac{1}{2}n(n-1)$, from which $n = 5$. The number of ways to select the three-person planning committee is $\binom{5}{3} = 10$.

2011B

11. **Answer (D):** If no more than 4 people have birthdays in any month, then at most 48 people would be accounted for. Therefore the statement is true for $n = 5$. The statement is false for $n \geq 6$ if, for example, 5 people have birthdays in each of the first 4 months of the year, and 4 people have birthdays in each of the last 8 months, for a total of $5 \cdot 4 + 4 \cdot 8 = 52$ people.

OR

The average number of birthdays per month is $\frac{52}{12}$, which is strictly between 4 and 5. Therefore at least one month must contain at least 5 birthdays, and, as above, it is possible to distribute the birthdays so that all months contain 4 or 5 birthdays.

2012B

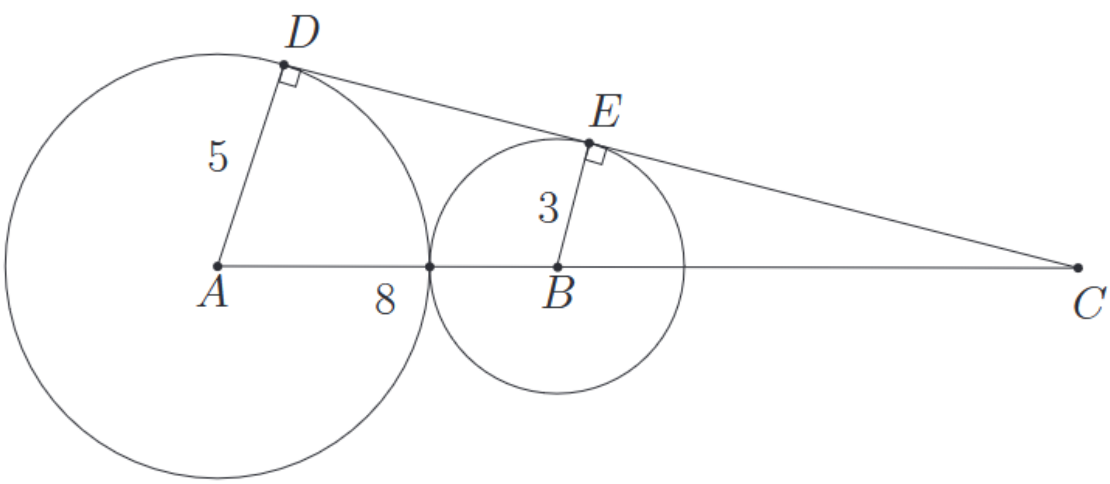
11. **Answer (D):** Let D and E be the points of tangency to circles A and B, respectively, of the common tangent line that intersects ray AB at point C. Then $AD = 5$, $BE = 3$, and $AB = 5 + 3 = 8$. Because right triangles ADC and BEC are similar, it follows that

$$\frac{BC}{AC} = \frac{BE}{AD},$$

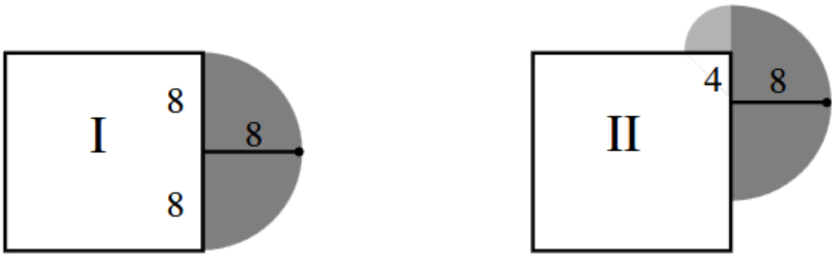
so

$$\frac{BC}{BC + 8} = \frac{3}{5}.$$

Solving gives $BC = 12$.



2006A 12. **(C)** The regions in which the dog can roam for each arrangement are shaded in the figure. For arrangement I, the area of this region is $\frac{1}{2}\pi \cdot 8^2 = 32\pi$ square feet. The area of the shaded region in arrangement II exceeds this by the area of a quarter-circle of radius 4 feet, that is, by $\frac{1}{4}\pi \cdot 4^2 = 4\pi$ square feet.



2004A 12. **(C)** A customer makes one of two choices for each condiment, to include it or not to include it. The choices are made independently, so there are $2^8 = 256$ possible combinations of condiments. For each of those combinations there are three choices regarding the number of meat patties, so there are altogether $(3)(256) = 768$ different kinds of hamburger.

2007A 12. **Answer (D):** The first guide can take any combination of tourists except all the tourists or none of the tourists. Therefore the number of possibilities is

$$\binom{6}{1}+\binom{6}{2}+\binom{6}{3}+\binom{6}{4}+\binom{6}{5}=6+15+20+15+6=62.$$

OR

If each guide did not need to take at least one tourist, then each tourist could choose one of the two guides independently. In this case there would be $2^6 = 64$ possible arrangements. The two arrangements for which all tourists choose the same guide must be excluded, leaving a total of $64 - 2 = 62$ possible arrangements.

2004A

13. **(D)** Because each man danced with exactly three women, there were $(12)(3) = 36$ pairs of men and women who danced together. Each woman had two partners, so the number of women who attended is $36/2 = 18$.

2011A

13. **Answer (A):** Because the numbers are even, they must end in either 2 or 8. If the last digit is 2, the first digit must be 5 and thus there are four choices remaining for the middle digit. If the last digit is 8, then there are two choices for the first digit, either 2 or 5, and for each choice there are four possibilities for the middle digit. The total number of choices is then $4 + 2 \cdot 4 = 12$.

2018B

13. **Answer (C):** The numbers in the given sequence are of the form $10^n + 1$ for $n = 2, 3, \ldots, 2019$. If n is even, say $n = 2k$ for some positive integer k, then $10^n + 1 = 100^k + 1 \equiv (-1)^k + 1 \pmod{101}$. Thus $10^n + 1$ is divisible by 101 if and only if k is odd, which means $n = 2, 6, 10, \ldots, 2018$. There are $\frac{1}{4}(2018 - 2) + 1 = 505$ such values. On the other hand, if n is odd, say $n = 2k + 1$ for some positive integer k, then

$$10^n + 1 = 10 \cdot 10^{n-1} + 1 = 10 \cdot 100^k + 1 \equiv 10 \cdot (-1)^k + 1 \pmod{101},$$

which is congruent to 9 or 11, and $10^n + 1$ is not divisible by 101 in this case.

2016A

12. **Answer (A):** The product of three integers is odd if and only if all three integers are odd. There are 1008 odd integers among the 2016 integers in the given range. The probability that all the selected integers are odd is

$$p = \frac{1008}{2016} \cdot \frac{1007}{2015} \cdot \frac{1006}{2014}.$$

The first factor is $\frac{1}{2}$ and each of the other factors is less than $\frac{1}{2}$, so $p < \frac{1}{8}$.

2002A

15. **(E)** The digits 2, 4, 5, and 6 cannot be the units digit of any two-digit prime, so these four digits must be the tens digits, and 1, 3, 7, and 9 are the units digits. The sum is thus

$$10(2 + 4 + 5 + 6) + (1 + 3 + 7 + 9) = 190.$$

(One set that satisfies the conditions is $\{23, 47, 59, 61\}$.)

2003B

15. (E) In the first round $100 - 64 = 36$ players are eliminated, one per match. In the second round there are 32 matches, in the third 16, then 8, 4, 2, and 1. The total number of matches is:

$$36 + 32 + 16 + 8 + 4 + 2 + 1 = 99.$$

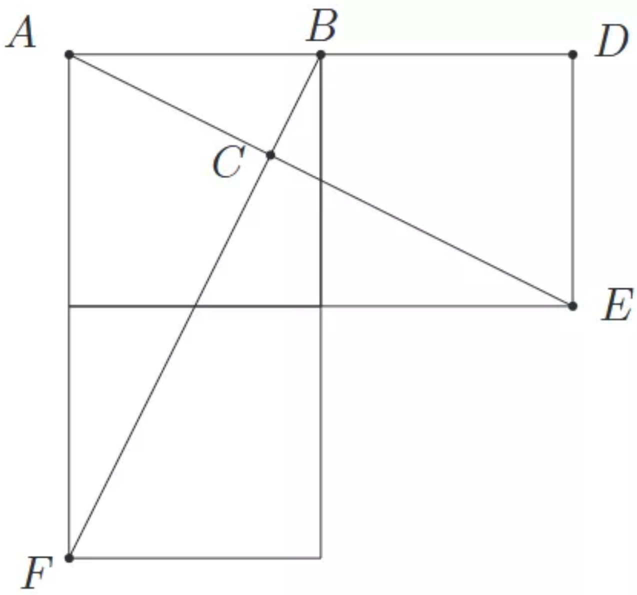
Note that 99 is divisible by 11, but 99 does not satisfy any of the other conditions given as answer choices.

OR

In each match, precisely one player is eliminated. Since there were 100 players in the tournament and all but one is eliminated, there must be 99 matches.

2012B

15. **Answer (B):** Place the figure on the coordinate plane with A at the origin, B on the positive x-axis, and label the other points as shown. Then the equation of line AE is $y = -\frac{1}{2}x$, and the equation of line BF is $y = 2x - 2$. Solving the simultaneous equations shows that $C = (\frac{4}{5}, -\frac{2}{5})$. Therefore $\triangle ABC$ has base $AB = 1$ and altitude $\frac{2}{5}$, so its area is $\frac{1}{5}$.



OR

Congruent right triangles AED and FBA have the property that their corresponding legs are perpendicular; hence their hypotenuses are perpendicular. Therefore $\angle ACB$ is a right angle and $\triangle ACB$ is similar to $\triangle FAB$. Because $AB = 1$ and $BF = \sqrt{5}$, the ratio of the area of $\triangle ACB$ to that of $\triangle FAB$ is 1 to 5. The area of $\triangle FAB$ is 1, so the area of $\triangle ACB$ is $\frac{1}{5}$.