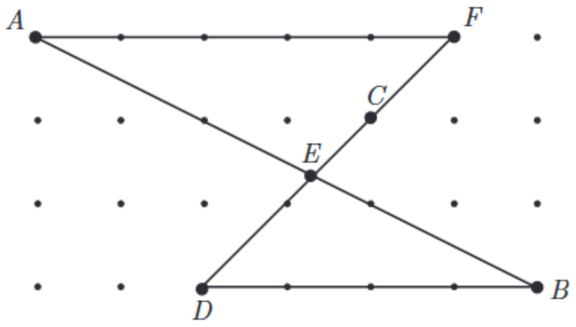
1

2D GEOMTERY

2000

16. **Answer (B):** Extend $\overline{DC}$ to F. Triangle FAE and DBE are similar with ratio $5:4$. Thus $AE = 5 \cdot AB/9$, $AB = \sqrt{3^2+6^2} = \sqrt{45} = 3\sqrt{5}$, and $AE = 5(3\sqrt{5})/9 = 5\sqrt{5}/3$.



OR

Coordinatize the points so that $A = (0,3)$, $B = (6,0)$, $C = (4,2)$, and $D = (2,0)$. Then the line through A and B is given by $x + 2y = 6$, and the line through C and D is given by $x - y = 2$. Solve these simultaneously to get $E = \left(\frac{10}{3}, \frac{4}{3}\right)$. Hence $AE = \sqrt{\left(\frac{10}{3}-0\right)^2 + \left(\frac{4}{3}-3\right)^2} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}$.

2004A

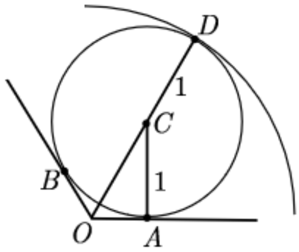
16. **(D)** All of the squares of size 5×5, 4×4, and 3×3 contain the black square and there are

$$1^2 + 2^2 + 3^2 = 14$$

of these. In addition, 4 of the 2×2 squares and 1 of the 1×1 squares contain the black square, for a total of $14 + 4 + 1 = 19$.

2004B

16. **(D)** Let O be the center of the large circle, let C be the center of one of the small circles, and let $\overline{OA}$ and $\overline{OB}$ be tangent to the small circle at A and B.



By symmetry, $\angle AOB = 120°$ and $\angle AOC = 60°$. Thus $\triangle AOC$ is a 30-60-90 degree right triangle, and $AC = 1$, so

$$OC = \frac{2}{\sqrt{3}}AC = \frac{2\sqrt{3}}{3}.$$

If OD is a radius of the large circle through C, then

$$OD = CD + OC = 1 + \frac{2\sqrt{3}}{3} = \frac{3+2\sqrt{3}}{3}.$$

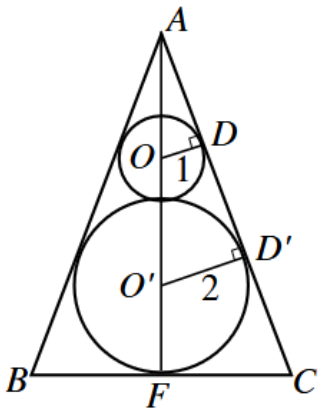
2006A

16. **(D)** Let O and O' denote the centers of the smaller and larger circles, respectively. Let D and D' be the points on $\overline{AC}$ that are also on the smaller and larger circles, respectively. Since $\triangle ADO$ and $\triangle AD'O'$ are similar right triangles, we have

$$\frac{AO}{1} = \frac{AO'}{2} = \frac{AO+3}{2}, \quad \text{so} \quad AO = 3.$$

As a consequence,

$$AD = \sqrt{AO^2 - OD^2} = \sqrt{9-1} = 2\sqrt{2}.$$



Let F be the midpoint of $\overline{BC}$. Since $\triangle ADO$ and $\triangle AFC$ are similar right triangles, we have

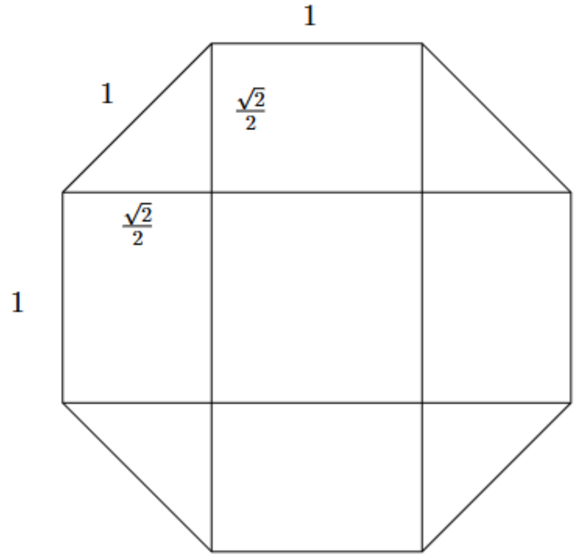
$$\frac{FC}{1} = \frac{AF}{AD} = \frac{AO + OO' + O'F}{AD} = \frac{3+3+2}{2\sqrt{2}} = 2\sqrt{2}.$$

So the area of $\triangle ABC$ is

$$\frac{1}{2} \cdot BC \cdot AF = \frac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}.$$

2011B

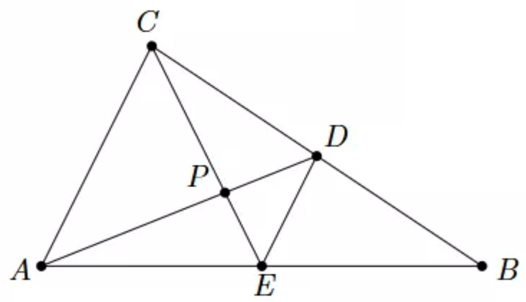
16. **Answer (A):** Assume the octagon's edge is 1. Then the corner triangles have hypotenuse 1 and thus legs $\frac{\sqrt{2}}{2}$ and area $\frac{1}{4}$ each; the four rectangles are 1 by $\frac{\sqrt{2}}{2}$ and have area $\frac{\sqrt{2}}{2}$ each, and the center square has area 1. The total area is $4\cdot\frac{1}{4}+4\cdot\frac{\sqrt{2}}{2}+1=2+2\sqrt{2}$. The probability that the dart hits the center square is $\frac{1}{2+2\sqrt{2}}=\frac{\sqrt{2}-1}{2}$.



2012B

16. **Answer (C):** Label the runners A, B, and C in increasing order of speed. After the start, runner B and runner C will be together again once runner C has run an extra 500 meters. Hence it takes $\frac{500}{5.0-4.8}=2500$ seconds for runners B and C to be together again. Similarly, it takes $\frac{500}{4.8-4.4}=1250$ seconds for runner A and runner B to be together again. Runners A and B will also be together at $2\cdot 1250=2500$ seconds, at which time all three runners will be together.

2013B **16. Answer (B):** The ratio of $PE{:}PD{:}DE$ is 3:4:5.



Hence by the converse of the Pythagorean Theorem, $\triangle DPE$ is a right triangle. Therefore $\overline{CE}$ is perpendicular to $\overline{AD}$, and the area of $AEDC$ is one-half the product of its diagonals. Because P is the centroid of $\triangle ABC$, it follows that $CE = 3(PE) = 4.5$ and $AD = 3(PD) = 6$. Therefore the area of $AEDC$ is $0.5(4.5)(6) = 13.5$.

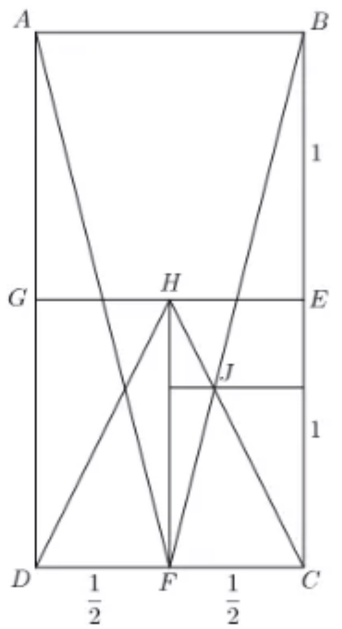
OR

From the first solution, triangles CPD, DPE, EPA, and APC are right triangles with right angle at P. The area of trapezoid $AEDC$ is given by the sum of the areas of these four triangles. Because $\overline{DE}$ is parallel to $\overline{AC}$ and D is the midpoint of $\overline{CB}$, triangles BAC and BED are similar with common ratio 2, so $AC = 2 \cdot DE = 5$. Triangles APC and DPE are similar, so $AP = 4$ and $CP = 3$. Thus the area of $AEDC$ is

$$\frac{1}{2} \cdot 4 \cdot 1.5 + \frac{1}{2} \cdot 3 \cdot 4 + \frac{1}{2} \cdot 2 \cdot 3 + \frac{1}{2} \cdot 2 \cdot 1.5 = 13.5\,.$$

2014A

16. **Answer (E):** Let J be the intersection point of $\overline{BF}$ and $\overline{HC}$. Then $\triangle JHF$ is similar to $\triangle JCB$ with ratio $1:2$. The length of the altitude of $\triangle JHF$ to $\overline{HF}$ plus the length of the altitude of $\triangle JCB$ to $\overline{CB}$ is $FC = \frac{1}{2}$. Thus $\triangle JHF$ has altitude $\frac{1}{6}$ and base 1, and its area is $\frac{1}{12}$. The shaded area is twice the area of $\triangle JHF$, or $\frac{1}{6}$.



OR

Place the figure on the coordinate plane with H at the origin. Then the equation of line DH is $y = 2x$, and the equation of line AF is $y = -4x - 1$. Solving the equations simultaneously shows that the leftmost point of the shaded region has x-coordinate $-\frac{1}{6}$. The kite therefore has diagonals $\frac{1}{3}$ and 1, so its area is $\frac{1}{2} \cdot \frac{1}{3} \cdot 1 = \frac{1}{6}$.

2001

17. **(C)** The slant height of the cone is 10, the radius of the sector. The circmference of the base of the cone is the same as the length of the secotr's arc. This is $252/360 = 7/10$ of the circumference, 20π, of the circle from which the sector is cut. The base circumference of the cone is 14π, so its radius is 7.

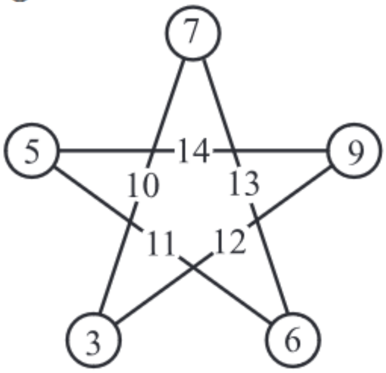
2005A

17. **(D)** Each number appears in two sums, so the sum of the sequence is

$$2(3+5+6+7+9)=60.$$

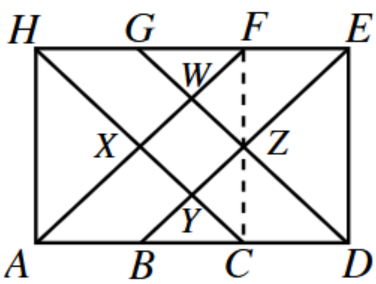
The middle term of a five-term arithmetic sequence is the mean of its terms, so $60/5=12$ is the middle term.

The figure shows an arrangement of the five numbers that meets the requirement.



2006A

17. **(A)** First note that since points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$, we have $HG=GF=FE=AB=BC=CD=1$. Also, $\overline{HG}$ is parallel to $\overline{CD}$ and $HG=CD$, so $CDGH$ is a parallelogram. Similarly, $\overline{AB}$ is parallel to $\overline{FE}$ and $AB=FE$, so $ABEF$ is a parallelogram. As a consequence, $WXYZ$ is a parallelogram, and since $HG=CD=AB=FE$, it is a rhombus.



Since $AH=AC=2$, the rectangle $ACFH$ is a square of side length 2. Its diagonals $\overline{AF}$ and $\overline{CH}$ have length $2\sqrt{2}$ and form a right angle at X. As a consequence, $WXYZ$ is a square. In isosceles $\triangle HXF$ we have $HX=XF=\sqrt{2}$. In addition, $HG=\frac{1}{2}HF$. So $XW=\frac{1}{2}XF=\frac{1}{2}\sqrt{2}$, and the square $WXYZ$ has area $XW^2=1/2$.

2009B 17. **Answer (C):** The area of the entire region is 5. The shaded region consists of a triangle with base $3-a$ and altitude 3, with one unit square removed. Therefore

$$\frac{3(3-a)}{2}-1=\frac{5}{2}.$$

Solving this equation yields $a=\frac{2}{3}$.

2011B 17. **Answer (C):** Angle EAB is 90° because it subtends a diameter. Therefore angles BEA and ABE are 40° and 50°, respectively. Angle DEB is 50° because $\overline{AB}$ is parallel to $\overline{ED}$. Also, $\angle DEB$ is supplementary to $\angle CDE$, so $\angle CDE = 130^\circ$. Because $\overline{EB}$ and $\overline{DC}$ are parallel chords, $ED = BC$ and $EBCD$ is an isosceles trapezoid. Thus $\angle BCD = \angle CDE = 130^\circ$.

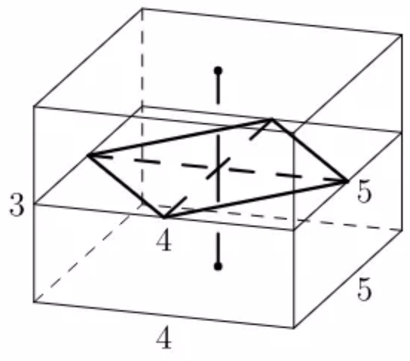
OR

Let O be the center of the circle. Establish, as in the first solution, that $\angle EAB = 90^\circ$, $\angle BEA = 40^\circ$, $\angle ABE = 50^\circ$, and $\angle DEB = 50^\circ$. Thus $\overline{AD}$ is a diameter and $\angle AOE = 100^\circ$. By the Inscribed Angle Theorem

$$\angle BCD = \frac{1}{2}(\angle BOA + \angle AOE + \angle EOD) = \frac{1}{2}(80^\circ + 100^\circ + 80^\circ) = 130^\circ\,.$$

2015B

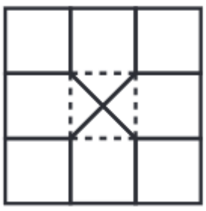
17. **Answer (B):** Consider the octahedron to be two pyramids whose base is a rhombus in the middle horizontal plane, as shown below. One pyramid points up, the other down. The area of the base is the area of 4 right triangles with legs 2 and $\frac{5}{2}$, or 10. The altitude of each pyramid is half that of the prism or $\frac{3}{2}$. The volume of the octahedron is $2 \cdot \frac{1}{3} \cdot 10 \cdot \frac{3}{2} = 10$.

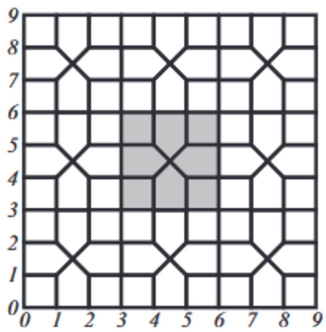


2001

18. **(D)** The pattern shown at left is repeated in the plane. In fact, nine repetitions of it are shown in the statement of the problem. Note that four of the nine squres in the three-by-three square are not in the four pentagons that make up the three-by-three square. Therefore, the percentage of the plane that is enclosed by pentagons is

$$1 - \frac{4}{9} = \frac{5}{9} = 55\frac{5}{9}\%$$



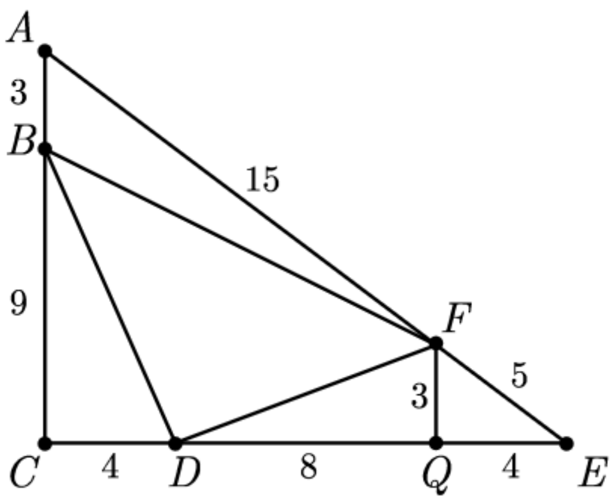


2004B 18. **(E)** The area of $\triangle ACE$ is $(1/2)(12)(16) = 96$. Draw $\overline{FQ} \perp \overline{CE}$. By similar triangles, $FQ = 3$ and $QE = 4$. The area of trapezoid $BCQF$ is $(1/2)(3+9)(12) = 72$. Since $\triangle BCD$ and $\triangle FDQ$ have areas 18 and 12, respectively, the area of $\triangle BDF$ is $72 - 18 - 12 = 42$. The desired ratio is $42/96 = 7/16$.

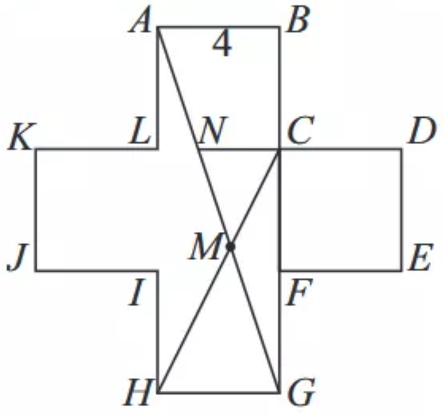
OR

Note that each of $\triangle ABF$, $\triangle BCD$, and $\triangle DEF$ has a base-altitude pair where the base and altitude are, respectively, $3/4$ and $1/4$ that of a corresponding base and altitude for $\triangle ACE$. Hence

$$\frac{\text{Area of} \triangle BDF}{\text{Area of} \triangle ACE} = 1 - 3(1/4)(3/4) = 7/16.$$



2007A 18. **Answer (C):** Extend $\overline{CD}$ past C to meet $\overline{AG}$ at N.



Since $\triangle ABG$ is similar to $\triangle NCG$,

$$NC = AB \cdot \frac{CG}{BG} = 4 \cdot \frac{8}{12} = \frac{8}{3}.$$

This implies that trapezoid $ABCN$ has area

$$\frac{1}{2} \cdot \left(\frac{8}{3} + 4\right) \cdot 4 = \frac{40}{3}.$$

Let v denote the length of the perpendicular from M to $\overline{NC}$. Since $\triangle CMN$ is similar to $\triangle HMG$, and

$$\frac{GH}{NC} = \frac{4}{8/3} = \frac{3}{2},$$

the length of the perpendicular from M to $\overline{HG}$ is $\frac{3}{2}v$. Because

$$v + \frac{3}{2}v = 8, \quad \text{we have} \quad v = \frac{16}{5}.$$

Hence the area of $\triangle CMN$ is

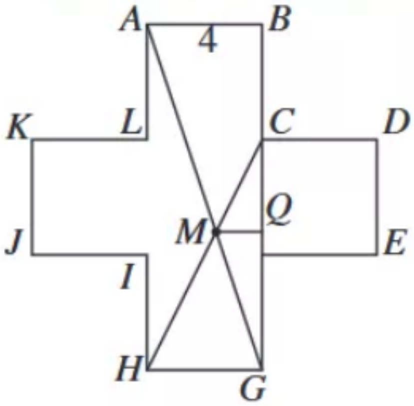
$$\frac{1}{2} \cdot \frac{8}{3} \cdot \frac{16}{5} = \frac{64}{15}.$$

So

$$\text{Area}(ABCM) = \text{Area}(ABCN) + \text{Area}(\triangle CMN) = \frac{40}{3} + \frac{64}{15} = \frac{88}{5}.$$

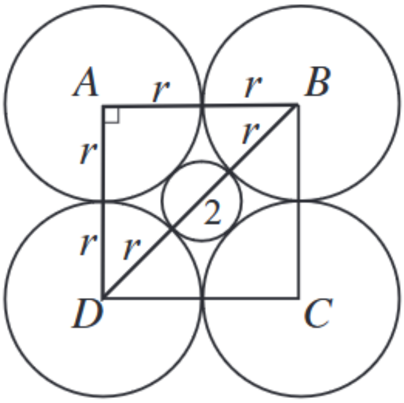
OR

Let Q be the foot of the perpendicular from M to $\overline{BG}$.



Since $\triangle MQG$ is similar to $\triangle ABG$, we have

2007B 18. **Answer (B):** Construct the square $ABCD$ by connecting the centers of the large circles, as shown, and consider the isosceles right $\triangle BAD$.



Since $AB = AD = 2r$ and $BD = 2 + 2r$, we have $2(2r)^2 = (2 + 2r)^2$. So

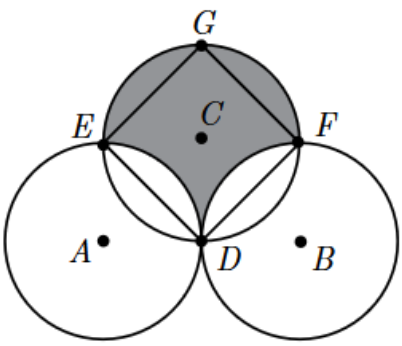
$$1 + 2r + r^2 = 2r^2, \quad \text{and} \quad r^2 - 2r - 1 = 0.$$

Applying the quadratic formula gives $r = 1 + \sqrt{2}$.

2011A 18. **Answer (C):** Let D be the midpoint of $\overline{AB}$, and let circle C intersect circles A and B at E and F, respectively, distinct from D. The shaded portion of unit square $ADCE$ has area $1 - \frac{\pi}{4}$, as does the shaded portion of unit square $BDCF$. The portion of the shaded region which is outside these squares is a semicircle of radius 1 and has area $\frac{\pi}{2}$. The total shaded area is $2\left(1 - \frac{\pi}{4}\right) + \frac{\pi}{2} = 2$.

OR

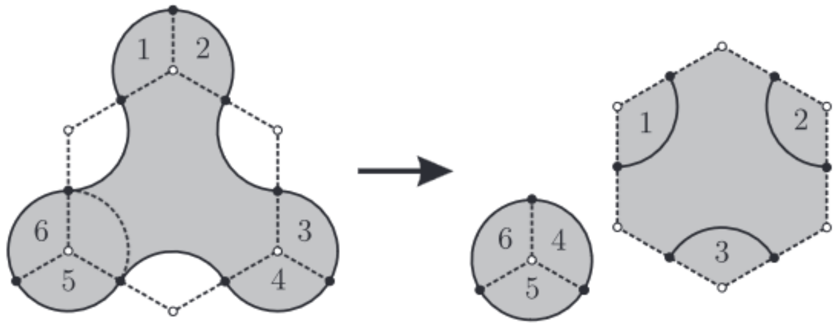
Let D, E, and F be defined as in the first solution, and let G be diametrically opposite D on circle C. The shaded area is equal to the area of square $DFGE$, which has diagonal length 2. Its side length is $\sqrt{2}$, and its area is $\left(\sqrt{2}\right)^2 = 2$.



2012A

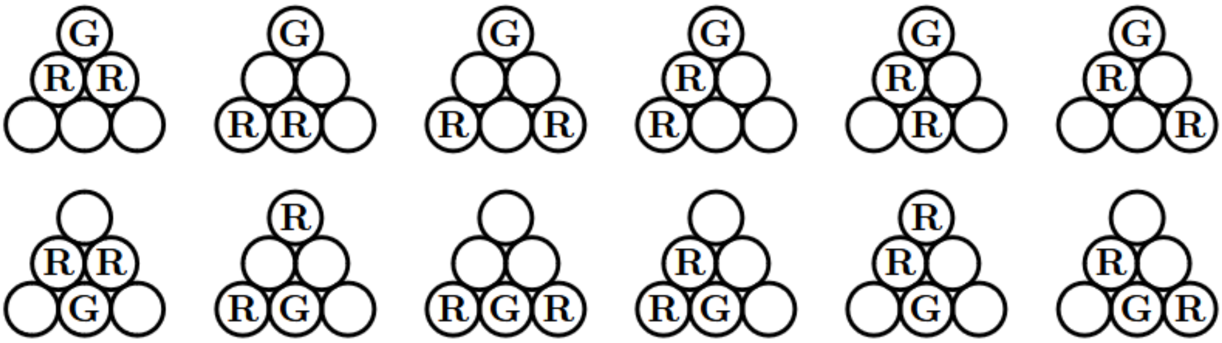
18. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4} \cdot 2^2\right) = \pi + 6\sqrt{3}\,.$$



2017B

18. **Answer (D):** By symmetry, there are just two cases for the position of the green disk: corner or non-corner. If a corner disk is painted green, then there is 1 case in which both red disks are adjacent to the green disk, there are 2 cases in which neither red disk is adjacent to the green disk, and there are 3 cases in which exactly one of the red disks is adjacent to the green disk. Similarly, if a non-corner disk is painted green, then there is 1 case in which neither red disk is in a corner, there are 2 cases in which both red disks are in a corner, and there are 3 cases in which exactly one of the red disks is in a corner. The total number of paintings is $1 + 2 + 3 + 1 + 2 + 3 = 12$.



2003A 19. (C) First note that the area of the region determined by the triangle topped by the semicircle of diameter 1 is

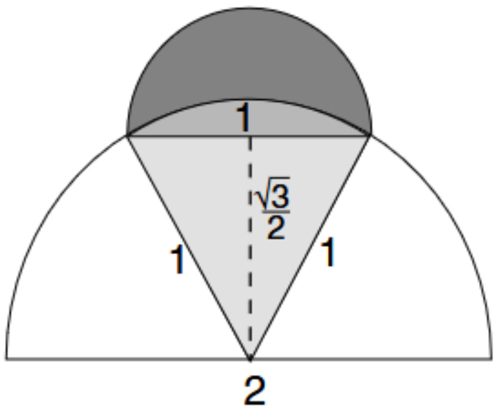
$$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{8}\pi.$$

The area of the lune results from subtracting from this the area of the sector of the larger semicircle,

$$\frac{1}{6}\pi(1)^2 = \frac{1}{6}\pi.$$

So the area of the lune is

$$\frac{\sqrt{3}}{4} + \frac{1}{8}\pi - \frac{1}{6}\pi = \frac{\sqrt{3}}{4} - \frac{1}{24}\pi.$$



Note that the answer does not depend on the position of the lune on the semicircle.

2003B

19. (E) The area of the larger semicircle is

$$\frac{1}{2}\pi(2)^2 = 2\pi.$$

The region deleted from the larger semicircle consists of five congruent sectors and two equilateral triangles. The area of each of the sectors is

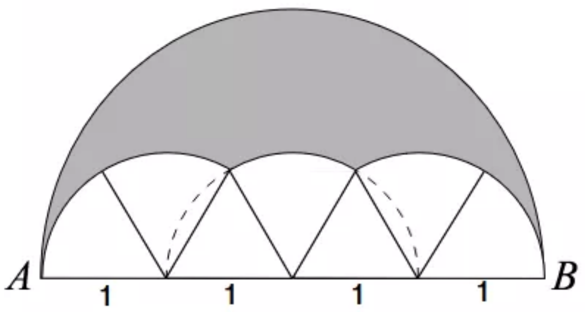
$$\frac{1}{6}\pi(1)^2 = \frac{\pi}{6}$$

and the area of each triangle is

$$\frac{1}{2}\cdot 1\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4},$$

so the area of the shaded region is

$$2\pi - 5\cdot\frac{\pi}{6} - 2\cdot\frac{\sqrt{3}}{4} = \frac{7}{6}\pi - \frac{\sqrt{3}}{2}.$$



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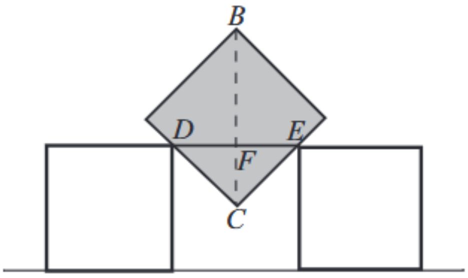
2005A 19. **(D)** Consider the rotated middle square shown in the figure. It will drop until length DE is 1 inch. Thus

$$FC = DF = FE = \frac{1}{2} \quad \text{and} \quad BC = \sqrt{2}.$$

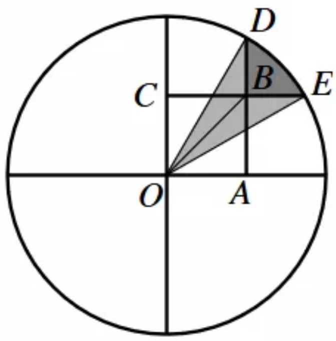
Hence $BF = \sqrt{2} - 1/2$. This is added to the 1 inch height of the supporting squares, so the overall height of point B above the line is

$$1 + BF = \sqrt{2} + \frac{1}{2} \text{ inches.}$$



2006B 19. **(A)** Since $OC = 1$ and $OE = 2$, it follows that $\angle EOC = 60°$ and $\angle EOA = 30°$. The area of the shaded region is the area of the $30°$ sector DOE minus the area of congruent triangles OBD and OBE. First note that

$$\text{Area(Sector } DOE) = \frac{1}{12}(4\pi) = \frac{\pi}{3}.$$



In right triangle OCE, we have $CE = \sqrt{3}$, so $BE = \sqrt{3} - 1$. Therefore

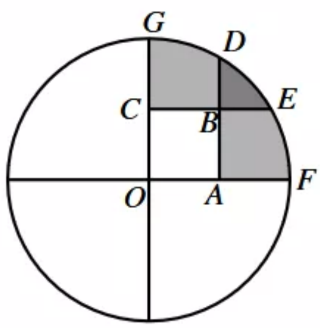
$$\text{Area}(\triangle OBE) = \frac{1}{2}(\sqrt{3} - 1)(1).$$

The required area is consequently

$$\frac{\pi}{3} - 2\left(\frac{\sqrt{3} - 1}{2}\right) = \frac{\pi}{3} + 1 - \sqrt{3}.$$

OR

Let F be the point where ray OA intersects the circle, and let G be the point where ray OC intersects the circle.



Let a be the area of the shaded region described in the problem, and b be the area of the region bounded by $\overline{AD}$, $\overline{AF}$, and the minor arc from D to F. Then b is also the area of the region bounded by $\overline{CE}$, $\overline{CG}$, and the minor arc from G to E. By the Inclusion-Exclusion Principle,

$$2b - a = \text{Area (Quartercircle } OFG) - \text{Area (Square } OABC) = \pi - 1.$$

Since b is the area of a $60°$ sector from which the area of $\triangle OAD$ has been deleted, we have

$$b = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

Hence the area of the shaded region described in the problem is

$$a = 2b - \pi + 1 = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \pi + 1 = \frac{\pi}{3} + 1 - \sqrt{3}.$$

2007A

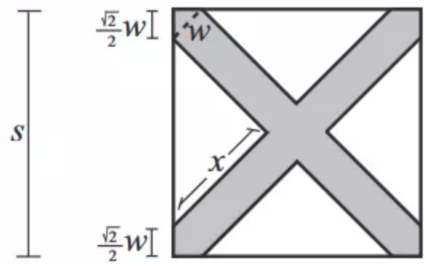
19. **Answer (C):** Let s be the side length of the square, let w be the width of the brush, and let x be the leg length of one of the congruent unpainted isosceles right triangles. Since the unpainted area is half the area of the square, the area of each unpainted triangle is 1/8 of the area of the square. So

$$\frac{1}{2}x^2 = \frac{1}{8}s^2 \quad \text{and} \quad x = \frac{1}{2}s.$$

The leg length x plus the brush width w is equal to half the diagonal of the square, so $x + w = (\sqrt{2}/2)s$. Thus

$$w = \frac{\sqrt{2}}{2}s - \frac{1}{2}s \quad \text{and} \quad \frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

OR



The painted stripes have isosceles right triangles with hypotenuse w at each vertex of the square, and the legs of these triangles have length $(\sqrt{2}/2)w$. Since the total area of the four congruent unpainted triangles is half the area of the original square, we have

$$s - \sqrt{2}w = \frac{s}{\sqrt{2}}, \quad \text{so} \quad \sqrt{2}s - 2w = s.$$

and

$$\frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

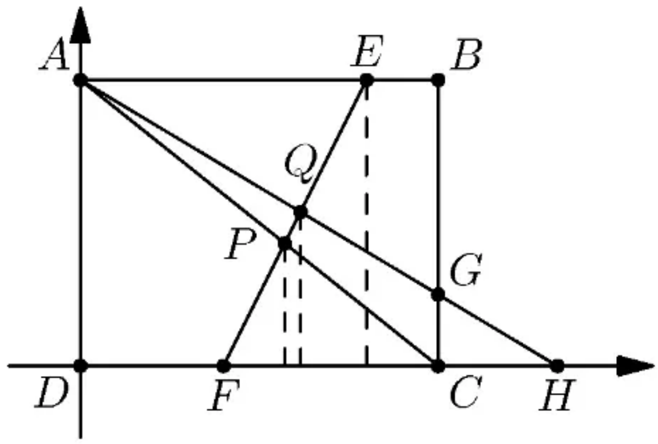
2007B

19. **Answer (C):** The first remainder is even with probability $2/6 = 1/3$ and odd with probability $2/3$. The second remainder is even with probability $3/6 = 1/2$ and odd with probability $1/2$. The shaded squares are those that indicate that both remainders are odd or both are even. Hence the square is shaded with probability

$$\frac{1}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{2}.$$

2016B

19. **Answer (D):** Triangles AEP and CFP are similar and $FP : EP = CF : AE = 3 : 4$, so $FP = \frac{3}{7}EF$. Extend $\overline{AG}$ and $\overline{FC}$ to meet at point H; then $\triangle AEQ$ and $\triangle HFQ$ are similar. Note that $\triangle HCG$ and $\triangle ABG$ are similar with sides in a ratio of $1 : 3$, so $CH = \frac{1}{3} \cdot 5$ and $FH = 3 + \frac{5}{3} = \frac{14}{3}$. Then $FQ : EQ = \frac{14}{3} : 4 = 7 : 6$, so $FQ = \frac{7}{13}FE$. Thus $PQ = FQ - FP = \left(\frac{7}{13} - \frac{3}{7}\right)FE = \frac{10}{91}FE$ and $\frac{PQ}{FE} = \frac{10}{91}$.



OR

Place the figure in the coordinate plane with D at the origin, A at $(0, 4)$, and C at $(5, 0)$. Then the equations of lines AC, AG, and EF are $y = -\frac{4}{5}x + 4$, $y = -\frac{3}{5}x + 4$, and $y = 2x - 4$, respectively. The intersections can be found by solving simultaneous linear equations: $P\left(\frac{20}{7}, \frac{12}{7}\right)$ and $Q\left(\frac{40}{13}, \frac{28}{13}\right)$. Because F, P, Q, and E are aligned, ratios of distances between these points are the same as ratios of the corresponding distances between their coordinates. Then

$$\frac{PQ}{FE} = \frac{\frac{40}{13} - \frac{20}{7}}{4 - 2} = \frac{10}{91}.$$

2002A

20. **(D)** Since $\triangle AGD$ is similar to $\triangle CHD$, we have $HC/1 = AG/3$. Also, $\triangle AGF$ is similar to $\triangle EJF$, so $JE/1 = AG/5$. Hence,

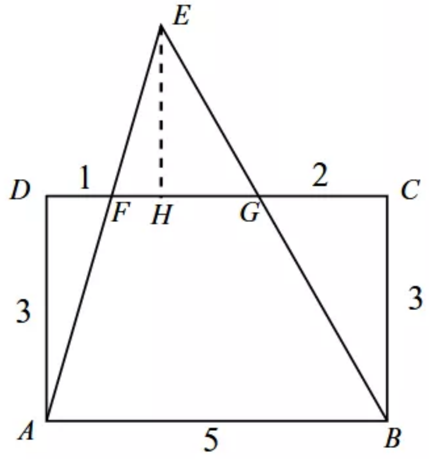
$$\frac{HC}{JE} = \frac{AG/3}{AG/5} = \frac{5}{3}.$$

2003B 20. (D) Let H be the foot of the perpendicular from E to $\overline{DC}$. Since $CD = AB = 5$, $FG = 2$, and $\triangle FEG$ is similar to $\triangle AEB$, we have

$$\frac{EH}{EH+3} = \frac{2}{5}, \quad \text{so} \quad 5EH = 2EH + 6,$$

and $EH = 2$. Hence

$$\text{Area}(\triangle AEB) = \frac{1}{2}(2+3) \cdot 5 = \frac{25}{2}.$$



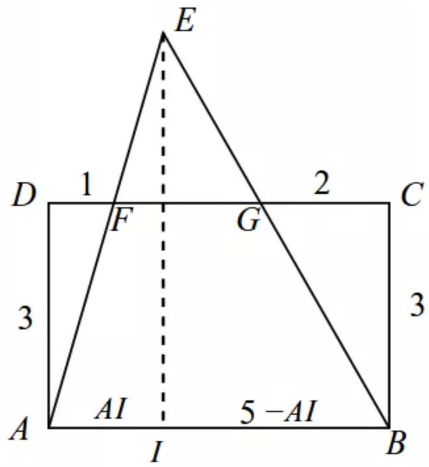
OR

Let I be the foot of the perpendicular from E to $\overline{AB}$. Since

$$\triangle EIA \text{ is similar to } \triangle ADF \quad \text{and} \quad \triangle EIB \text{ is similar to } \triangle BCG,$$

we have

$$\frac{AI}{EI} = \frac{1}{3} \quad \text{and} \quad \frac{5 - AI}{EI} = \frac{2}{3}.$$



Adding gives $5/EI = 1$, so $EI = 5$. The area of the triangle is $\frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

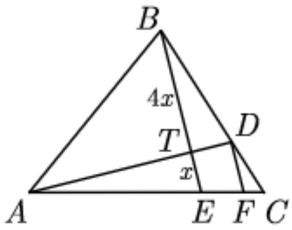
2004A

20. **(D)** First, assume that $AB = 1$, and let $ED = DF = x$. By the Pythagorean Theorem $x^2 + x^2 = EF^2 = EB^2 = 1^2 + (1-x)^2$, so $x^2 = 2(1-x)$. Hence the desired ratio of the areas is

$$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABE)} = \frac{x^2}{1-x} = 2.$$

2004B

20. **(D)** Let F be a point on $\overline{AC}$ such that $\overline{DF}$ is parallel to $\overline{BE}$. Let $BT = 4x$ and $ET = x$.



Because $\triangle ATE$ and $\triangle ADF$ are similar, we have

$$\frac{DF}{x} = \frac{AD}{AT} = \frac{4}{3}, \quad \text{and} \quad DF = \frac{4x}{3}.$$

Also, $\triangle BEC$ and $\triangle DFC$ are similar, so

$$\frac{CD}{BC} = \frac{DF}{BE} = \frac{4x/3}{5x} = \frac{4}{15}.$$

Thus

$$\frac{CD}{BD} = \frac{CD/BC}{1-(CD/BC)} = \frac{4/15}{1-4/15} = \frac{4}{11}.$$

OR

Let $s = \text{Area}(\triangle ABC)$. Then

$$\text{Area}(\triangle TBC) = \frac{1}{4}s \quad \text{and} \quad \text{Area}(\triangle ATC) = \frac{1}{5}s,$$

so

$$\text{Area}(\triangle ATB) = \text{Area}(\triangle ABC) - \text{Area}(\triangle TBC) - \text{Area}(\triangle ATC) = \frac{11}{20}s.$$

Hence

$$\frac{CD}{BD} = \frac{\text{Area}(\triangle ADC)}{\text{Area}(\triangle ABD)} = \frac{\text{Area}(\triangle ATC)}{\text{Area}(\triangle ATB)} = \frac{s/5}{11s/20} = \frac{4}{11}.$$

2009B

20. **Answer (B):** By the Pythagorean Theorem, $AC = \sqrt{5}$. By the Angle Bisector Theorem, $\frac{BD}{AB} = \frac{CD}{AC}$. Therefore $CD = \sqrt{5} \cdot BD$ and $BD + CD = 2$, from which

$$BD = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

OR

Let $\overline{DE}$ be an altitude of $\triangle ADC$. Then note that $\triangle ABD$ is congruent to $\triangle AED$, and so $AE = 1$. As in the first solution $AC = \sqrt{5}$. Let $x = BD$. Then $DE = x$, $EC = \sqrt{5} - 1$, and $DC = 2 - x$. Applying the Pythagorean Theorem to $\triangle DEC$ yields $x^2 + \left(\sqrt{5} - 1\right)^2 = (2 - x)^2$, from which $x = \frac{\sqrt{5}-1}{2}$.