11

PROBABILITY

2007A 16. **Answer (E):** The number $ad - bc$ is even if and only if ad and bc are both odd or are both even. Each of ad and bc is odd if both of its factors are odd, and even otherwise. Exactly half of the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $(1/2) \cdot (1/2) = 1/4$ and are even with probability $3/4$. Hence the probability that $ad - bc$ is even is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{5}{8}.$$

2008B 16. **Answer (A):** If one die is rolled, 3 of the 6 possible numbers are odd. If two dice are rolled, 18 of the 36 possible outcomes have odd sums. In each of these cases, the probability of an odd sum is $\frac{1}{2}$. If no die is rolled, the sum is 0, which is not odd. The probability that no die is rolled is equal to the probability that both coin tosses are tails, which is $(\frac{1}{2})^2 = \frac{1}{4}$. Thus the requested probability is

$$\left(1 - \frac{1}{4}\right) \cdot \frac{1}{2} = \frac{3}{8}.$$

2014B

16. **Answer (B):** If exactly three of the four dice show the same number, then there are 6 possible choices for the repeated value and 5 possible choices for the non-repeated value. The non-repeated value may appear on any one of the 4 dice, so there are $6 \cdot 5 \cdot 4 = 120$ possible ways for such a result to occur. There are 6 ways for all four dice to show the same value. There are 6^4 total possible outcomes for the four dice. The probability of the desired result is $\frac{120+6}{6^4} = \frac{7}{72}$.

2015B

16. **Answer (C):** There are 9 assignments satisfying the condition: $(4,2,1)$, $(6,2,1)$, $(8,2,1)$, $(10,2,1)$, $(6,3,1)$, $(9,3,1)$, $(8,4,1)$, $(10,5,1)$, and $(8,4,2)$. There are $10 \cdot 9 \cdot 8 = 720$ possible assignments, so the probability is $\frac{9}{720} = \frac{1}{80}$.

2006B

17. **(D)** After Alice puts the ball into Bob's bag, his bag will contain six balls: two of one color and one of each of the other colors. After Bob selects a ball and places it into Alice's bag, the two bags will have the same contents if and only if Bob picked one of the two balls in his bag that are the same color. Because there are six balls in the bag when Bob makes his selection, the probability of selecting one of the same colored pair is $2/6 = 1/3$.

2008B

17. **Answer (B):** The responses on these three occasions, in order, must be YNN, NYN, or NNY, where Y indicates approval and N indicates disapproval. The probability of each of these is $(0.7)(0.3)(0.3) = 0.063$, so the requested probability is $3(0.063) = 0.189$.

2014A

17. **Answer (D):** Each roll of the three dice can be recorded as an ordered triple (a, b, c) of the three values appearing on the dice. There are 6^3 equally likely triples possible. For the sum of two of the values in the triple to equal the third value, the triple must be a permutation of one of the triples $(1, 1, 2)$, $(1, 2, 3)$, $(1, 3, 4)$, $(1, 4, 5)$, $(1, 5, 6)$, $(2, 2, 4)$, $(2, 3, 5)$, $(2, 4, 6)$, or $(3, 3, 6)$. There are $3! = 6$ permutations of the values (a, b, c) when a, b, and c are distinct, and 3 permutations of the values when two of the values are equal. Thus there are $6 \cdot 6 + 3 \cdot 3 = 45$ triples where the sum of two of the values equals the third. The requested probability is $\frac{45}{6^3} = \frac{5}{24}$.

OR

There are 36 outcomes when a pair of dice are rolled, and the probability of rolling a total of $2, 3, 4, 5$, or 6 is $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}$, and $\frac{5}{36}$, respectively. The probability that another die matches this total is $\frac{1}{6}$, and there are 3 ways to choose the die that matches the total of the other two. Thus the requested probability is $3(\frac{1}{36} \cdot \frac{1}{6} + \frac{2}{36} \cdot \frac{1}{6} + \frac{3}{36} \cdot \frac{1}{6} + \frac{4}{36} \cdot \frac{1}{6} + \frac{5}{36} \cdot \frac{1}{6}) = 3 \cdot \frac{15}{36} \cdot \frac{1}{6} = \frac{5}{24}$.

2016A

14. **Answer (C):** If the sum uses n twos and m threes, then $2n + 3m = 2016$. Therefore $n = \frac{2016-3m}{2}$. Both m and n will be nonnegative integers if and only if m is an even integer from 0 to 672. Thus there are $\frac{672}{2} + 1 = 337$ ways to form the sum.

2005A **18. (A)**

There are four possible outcomes,

$$\text{ABAA}, \quad \text{ABABA}, \quad \text{ABBAA}, \quad \text{and} \quad \text{BBAAA},$$

but they are not equally likely. This is because, in general, the probability of any specific four-game series is $(1/2)^4 = 1/16$, whereas the probability of any specific five-game series is $(1/2)^4 = 1/32$. Thus the first listed outcome is twice as likely as each of the other three. Let p be the probability of the occurrence $ABBAA$. Then the probability of $ABABA$ is also p, as is the probability of $BBAAA$, whereas the probability of $ABAA$ is $2p$. So

$$2p + p + p + p = 1, \quad \text{and} \quad p = \frac{1}{5}.$$

The only outcome in which team B wins the first game is $BBAAA$, so the probability of this occurring is $1/5$.

OR

To consider equally-likely cases, suppose that all five games are played, even if team A has won the series before the fifth game. Then the possible ways that team A can win the series, given that team B wins the second game, are

$$\text{BBAAA}, \quad \text{ABBAA}, \quad \text{ABABA}, \quad \text{ABAAB}, \quad \text{and} \quad \text{ABAAA}.$$

In only the first case does team B win the first game, so the probability of this occurring is $1/5$.

2010A

18. **Answer (B):** The probability that Bernardo picks a 9 is $\frac{3}{9} = \frac{1}{3}$. In this case, his three-digit number will begin with a 9 and will be larger than Silvia's three-digit number.

If Bernardo does not pick a 9, then Bernardo and Silvia will form the same number with probability

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

If they do not form the same number then Bernardo's number will be larger $\frac{1}{2}$ of the time.

Hence the probability is

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}(1 - \frac{1}{56}) = \frac{111}{168} = \frac{37}{56}.$$

2010B

18. **Answer (E):** Let $N = abc + ab + a = a(bc + b + 1)$. If a is divisible by 3, then N is divisible by 3. Note that 2010 is divisible by 3, so the probability that a is divisible by 3 is $\frac{1}{3}$.

If a is not divisible by 3 then N is divisible by 3 if $bc + b + 1$ is divisible by 3. Define b_0 and b_1 so that $b = 3b_0 + b_1$ is an integer and b_1 is equal to 0, 1, or 2. Note that each possible value of b_1 is equally likely. Similarly define c_0 and c_1. Then

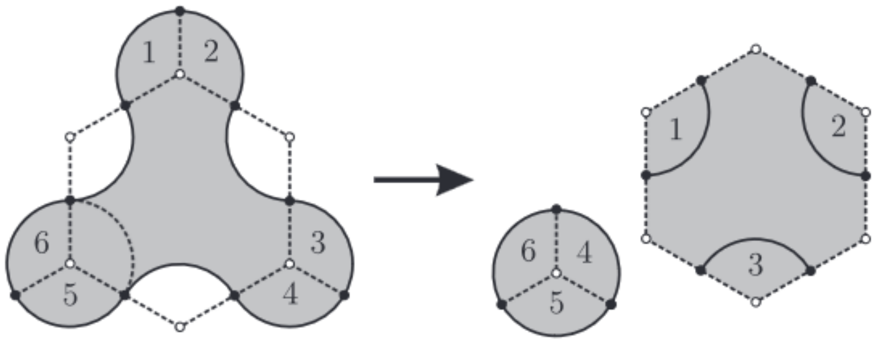
$$\begin{aligned} bc + b + 1 &= (3b_0 + b_1)(3c_0 + c_1) + 3b_0 + b_1 + 1 \\ &= 3(3b_0c_0 + c_0b_1 + c_1b_0 + b_0) + b_1c_1 + b_1 + 1. \end{aligned}$$

Hence $bc + b + 1$ is divisible by 3 if and only if $b_1 = 1$ and $c_1 = 1$, or $b_1 = 2$ and $c_1 = 0$. The probability of this occurrence is $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.

Therefore the requested probability is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$.

2012B 18. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4} \cdot 2^2\right) = \pi + 6\sqrt{3}.$$



2014B

18. **Answer (E):** The numbers in the list have a sum of $11 \cdot 10 = 110$. The value of the 11th number is maximized when the sum of the first ten numbers is minimized subject to the following conditions.

- If the numbers are arranged in nondecreasing order, the sixth number is 9.
- The number 8 occurs either 2, 3, 4, or 5 times, and all other numbers occur fewer times.

If 8 occurs 5 times, the smallest possible sum of the first 10 numbers is

$$8+8+8+8+8+9+9+9+9+10=86.$$

If 8 occurs 4 times, the smallest possible sum of the first 10 numbers is

$$1+8+8+8+8+9+9+9+10+10=80.$$

If 8 occurs 3 times, the smallest possible sum of the first 10 numbers is

$$1+1+8+8+8+9+9+10+10+11=75.$$

If 8 occurs 2 times, the smallest possible sum of the first 10 numbers is

$$1+2+3+8+8+9+10+11+12+13=77.$$

Thus the largest possible value of the 11th number is $110-75=35$.

2015B

18. **Answer (D):** A coin can be tossed once, twice, or three times. View the problem as tossing each coin three times. If all three tosses are tails then the coin ends on a tail; however, if any of the three tosses is a head then the coin ends on a head (the subsequent tosses can be ignored). Thus each coin has a 7 out of 8 chance of landing on heads. Therefore the expected number of heads is $\frac{7}{8} \cdot 64 = 56$.

2017A 18. **Answer (D):** Let x be the probability that Amelia wins. Then $x = \frac{1}{3} + \left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{5}\right)x$, because either Amelia wins on the first toss, or, if she and Blaine both get tails, then the chance of her winning from that point onward is also x. Solving this equation gives $x = \frac{5}{9}$. The requested difference is $9 - 5 = 4$.

OR

The probability that Amelia wins on the first toss is $\frac{1}{3}$, the probability that Amelia wins on the second toss is $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3}$, and so on. Therefore the probability that Amelia wins is

$$\begin{aligned}\frac{1}{3} + \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{5}\right)^2 \cdot \frac{1}{3} + \cdots &= \frac{1}{3} \cdot \left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots\right)\\ &= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{5}}\\ &= \frac{5}{9}.\end{aligned}$$

OR

Let $n \geq 0$ be the greatest integer such that Amelia and Blaine each toss tails n times in a row. Then the game will end in the next round of tosses, either because Amelia tosses a head, which will occur with probability $\frac{1}{3}$, or because Amelia tosses a tail and Blaine tosses a head, which will occur with probability $\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$. The probability that it is Amelia who wins is therefore

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{4}{15}} = \frac{5}{9}.$$

2018A 19. **Answer (E):** For $m \in \{11, 13, 15, 17, 19\}$, let $p(m)$ denote the probability that m^n has units digit 1, where n is chosen at random from the set $S = \{1999, 2000, 2001, \ldots, 2018\}$. Then the desired probability is equal to $\frac{1}{5}\big(p(11) + p(13) + p(15) + p(17) + p(19)\big)$. Because any positive integral power of 11 always has units digit 1, $p(11) = 1$, and because any positive integral power of 15 always has units digit 5, $p(15) = 0$. Note that S has 20 elements, exactly 5 of which are congruent to j mod 4 for each of $j = 0, 1, 2, 3$. The units digits of powers of 13 and 17 cycle in groups of 4. More precisely,

$$\left(13^k \bmod 10\right)_{k=1999}^{2018} = (7, 1, 3, 9, 7, 1, \ldots, 3, 9)$$

and

$$\left(17^k \bmod 10\right)_{k=1999}^{2018} = (3, 1, 7, 9, 3, 1, \ldots, 7, 9).$$

Thus $p(13) = p(17) = \frac{5}{20} = \frac{1}{4}$. Finally, note that the units digit of 19^k is 1 or 9, according to whether k is even or odd, respectively. Thus $p(19) = \frac{1}{2}$. Hence the requested probability is

$$\frac{1}{5}\left(1 + \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{2}\right) = \frac{2}{5}.$$

2006A 20. **(E)** Place each of the integers in a pile based on the remainder when the integer is divided by 5. Since there are only 5 piles but there are 6 integers, at least one of the piles must contain two or more integers. The difference of two integers in the same pile is divisible by 5. Hence the probability is 1.

We have applied what is called the Pigeonhole Principle. This states that if you have more pigeons than boxes and you put each pigeon in a box, then at least one of the boxes must have more than one pigeon. In this problem the pigeons are integers and the boxes are piles.

2008B 20. **Answer (B):** Of the 36 possible outcomes, the four pairs $(1,4)$, $(2,3)$, $(2,3)$, and $(4,1)$ yield a sum of 5. The six pairs $(1,6)$, $(2,5)$, $(2,5)$, $(3,4)$, $(3,4)$, and $(4,3)$ yield a sum of 7. The four pairs $(1,8)$, $(3,6)$, $(3,6)$, and $(4,5)$ yield a sum of 9. Thus the probability of getting a sum of 5, 7, or 9 is $(4+6+4)/36 = 7/18$.

Note: The dice described here are known as Sicherman dice. The probability of obtaining each sum between 2 and 12 is the same as that on a pair of standard dice.

2012A 20. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of $BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

2014B

19. **Answer (D):** Let A be the first point chosen on the outer circle, let chords $\overline{AB}$ and $\overline{AC}$ on the outer circle be tangent to the inner circle at D and E, respectively, and let O be the common center of the two circles. Triangle ADO has a right angle at D, $OA = 2$, and $OD = 1$, so $\angle OAD = 30°$. Similarly, $\angle OAE = 30°$, so $\angle BAC = \angle DAE = 60°$, and minor arc $BC = 120°$. If X is the second point chosen on the outer circle, then chord $\overline{AX}$ intersects the inner circle if and only if X is on minor arc BC. Therefore the requested probability is $\frac{120°}{360°} = \frac{1}{3}$.

2017B

20. **Answer (B):** There are $\lfloor\frac{21}{2}\rfloor+\lfloor\frac{21}{4}\rfloor+\lfloor\frac{21}{8}\rfloor+\lfloor\frac{21}{16}\rfloor = 10+5+2+1 = 18$ powers of 2 in the prime factorization of 21!. Thus $21! = 2^{18}k$, where k is odd. A divisor of 21! must be of the form $2^i b$ where $0 \le i \le 18$ and b is a divisor of k. For each choice of b, there is one odd divisor of 21! and 18 even divisors. Therefore the probability that a randomly chosen divisor is odd is $\frac{1}{19}$. In fact, $21! = 2^{18}\cdot 3^9\cdot 5^4\cdot 7^3\cdot 11\cdot 13\cdot 17\cdot 19$, so it has $19\cdot 10\cdot 5\cdot 4\cdot 2\cdot 2\cdot 2\cdot 2 = 60{,}800$ positive integer divisors, of which $10\cdot 5\cdot 4\cdot 2\cdot 2\cdot 2\cdot 2 = 3{,}200$ are odd.