15

SOLVE FOR X

2009A

2002A 16. **(B)** From the given information,

$$(a+1)+(b+2)+(c+3)+(d+4)=4(a+b+c+d+5),$$

so

$$(a+b+c+d)+10=4(a+b+c+d)+20$$

and $a+b+c+d=-\frac{10}{3}$.

OR

Note that $a=d+3$, $b=d+2$, and $c=d+1$. So,

$$a+b+c+d=(d+3)+(d+2)+(d+1)+d=4d+6.$$

Thus, $d+4=(4d+6)+5$, so $d=-7/3$, and

$$a+b+c+d=4d+6=4\left(-\frac{7}{3}\right)+6=-\frac{10}{3}.$$

2005B 17. **(B)** Because

$$4^{a\cdot b\cdot c\cdot d}=\left(\left(\left(4^a\right)^b\right)^c\right)^d=\left(\left(5^b\right)^c\right)^d=\left(6^c\right)^d=7^d=8=4^{3/2},$$

we have $a\cdot b\cdot c\cdot d=3/2$.

2007A 17. **Answer (D):** An integer is a cube if and only if, in the prime factorization of the number, each prime factor occurs a multiple of three times. Because $n^3=75m=3\cdot 5^2\cdot m$, the minimum value for m is $3^2\cdot 5=45$. In that case $n=15$, and $m+n=60$.

2012A

17. **Answer (C):** Note that

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{a^2 + ab + b^2}{a^2 - 2ab + b^2}.$$

Hence the given equation may be written as $3a^2+3ab+3b^2 = 73a^2-146ab+73b^2$. Combining like terms and factoring gives $(10a-7b)(7a-10b) = 0$. Because $a > b$, and a and b are relatively prime, $a = 10$ and $b = 7$. Thus $a - b = 3$.

2003B

18. (D) Among five consecutive odd numbers, at least one is divisible by 3 and exactly one is divisible by 5, so the product is always divisible by 15. The cases $n = 2$, $n = 10$, and $n = 12$ demonstrate that no larger common divisor is possible, since 15 is the greatest common divisor of $3\cdot5\cdot7\cdot9\cdot11$, $11\cdot13\cdot15\cdot17\cdot19$, and $13 \cdot 15 \cdot 17 \cdot 19 \cdot 21$.

2000

20. **Answer (C):** Note that

$$AMC+AM+MC+CA = (A+1)(M+1)(C+1)-(A+M+C)-1 = pqr-11,$$

where p, q, and r are positive integers whose sum is 13. A case-by-case analysis shows that pqt is largest when two of the numbers p, q, r are 4 and the third is 5. Thus the answer is $4 \cdot 4 \cdot 5 - 11 = 69$.

2002B

20. **(B)** We have $a + 8c = 4 + 7b$ and $8a - c = 7 - 4b$. Squaring both equations and adding the results yields

$$(a + 8c)^2 + (8a - c)^2 = (4 + 7b)^2 + (7 - 4b)^2.$$

Expanding gives $65(a^2+c^2) = 65(1+b^2)$. So $a^2+c^2 = 1+b^2$, and $a^2-b^2+c^2 = 1$.

2018B

20. **Answer (B):** Applying the recursion for several steps leads to the conjecture that

$$f(n) = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod 6, \\ n & \text{if } n \equiv 1 \pmod 6, \\ n-1 & \text{if } n \equiv 2 \pmod 6, \\ n & \text{if } n \equiv 3 \pmod 6, \\ n+2 & \text{if } n \equiv 4 \pmod 6, \\ n+3 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

The conjecture can be verified using the strong form of mathematical induction with two base cases and six inductive steps. For example, if $n \equiv 2 \pmod 6$, then $n = 6k+2$ for some nonnegative integer k and

$$\begin{aligned} f(n) &= f(6k+2) \\ &= f(6k+1) - f(6k) + 6k + 2 \\ &= (6k+1) - (6k+2) + 6k + 2 \\ &= 6k+1 \\ &= n-1. \end{aligned}$$

Therefore $f(2018) = f(6 \cdot 336 + 2) = 2018 - 1 = 2017$.

OR

Note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + (n-1)] - f(n-2) + n \\ &= -[f(n-4) - f(n-5) + (n-3)] + 2n - 1 \\ &= -[f(n-5) - f(n-6) + (n-4)] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

It follows that $f(2018) = f(2) + 2016 = 2017$.