16

ALGEBRA WORD PROBLEMS

2013A

17. **Answer (B):** Alice and Beatrix will visit Daphne together every $3 \cdot 4 = 12$ days, so this will happen $\lfloor \frac{365}{12} \rfloor = 30$ times. Likewise Alice and Claire will visit together $\lfloor \frac{365}{3 \cdot 5} \rfloor = 24$ times, and Beatrix and Claire will visit together $\lfloor \frac{365}{4 \cdot 5} \rfloor = 18$ times. However, each of these counts includes the $\lfloor \frac{365}{3 \cdot 4 \cdot 5} \rfloor = 6$ times when all three friends visit. The number of days that exactly two friends visit is $(30 - 6) + (24 - 6) + (18 - 6) = 54$.

2013B

17. **Answer (E):** After Alex makes m exchanges at the first booth and n exchanges at the second booth, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. No more exchanges are possible when he has fewer than 2 red tokens and fewer than 3 blue tokens. Therefore no more exchanges are possible if and only if $2m - n \geq 74$ and $3n - m \geq 73$. Equality can be achieved when $(m, n) = (59, 44)$, and Alex will have $59 + 44 = 103$ silver tokens.

Note that the following exchanges produce 103 silver tokens:

	Red Tokens	Blue Tokens	Silver Tokens
Exchange 75 blue tokens	100	0	25
Exchange 100 red tokens	0	50	75
Exchange 48 blue tokens	16	2	91
Exchange 16 red tokens	0	10	99
Exchange 9 blue tokens	3	1	102
Exchange 2 red tokens	1	2	103

2017B

17. **Answer (B):** The monotonous positive integers with one digit or increasing digits can be put into a one-to-one correspondence with the nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $2^9 - 1 = 511$. The monotonous positive integers with one digit or decreasing digits can be put into a one-to-one correspondence

with the subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ other than $\emptyset$ and $\{0\}$. The number of these is $2^{10} - 2 = 1022$. The single-digit numbers are included in both sets, so there are $511 + 1022 - 9 = 1524$ monotonous positive integers.

2015A

18. **Answer (E):** Because $1000 = 3 \cdot 16^2 + 14 \cdot 16 + 8$, the largest number less than 1000 whose hexadecimal representation contains only numeric digits is $3 \cdot 16^2 + 9 \cdot 16 + 9$. Thus the number of such positive integers is $n = 4 \cdot 10 \cdot 10 - 1 = 399$ $(0 \cdot 16^2 + 0 \cdot 16 + 0 = 0$ is excluded), and the sum of the digits of n is 21.

2013A

19. **Answer (C):** For the base-b representation of 2013 to end in the digit 3, the base b must exceed 3. Also, b must divide $2013 - 3 = 2010$, so b must be one of the 16 positive integer factors of $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Thus there are $16 - 3 = 13$ bases in which 2013 ends with a 3.

2003A

20. (E) The largest base-9 three-digit number is $9^3 - 1 = 728$ and the smallest base-11 three-digit number is $11^2 = 121$. There are 608 integers that satisfy $121 \le n \le 728$, and 900 three-digit numbers altogether, so the probability is $608/900 \approx 0.7$.

2012B

20. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of

$BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

2016A 17. **Answer (A):** Let $N = 5k$, where k is a positive integer. There are $5k + 1$ equally likely possible positions for the red ball in the line of balls. Number these $0, 1, 2, 3, \ldots, 5k - 1, 5k$ from one end. The red ball will *not* divide the green balls so that at least $\frac{3}{5}$ of them are on the same side if it is in position $2k + 1, 2k + 2, \ldots, 3k - 1$. This includes $(3k - 1) - 2k = k - 1$ positions. The probability that $\frac{3}{5}$ or more of the green balls will be on the same side is therefore $1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}$.

Solving the inequality $\frac{4k+2}{5k+1} < \frac{321}{400}$ for k yields $k > \frac{479}{5} = 95\frac{4}{5}$. The value of k corresponding to the required least value of N is therefore 96, so $N = 480$. The sum of the digits of N is 12.