17

RATIO

2003B

17. (B) Let r be the radius of the sphere and cone, and let h be the height of the cone. Then the conditions of the problem imply that

$$\frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^2 h, \quad \text{so } h = 3r.$$

Therefore, the ratio of h to r is $3:1$.

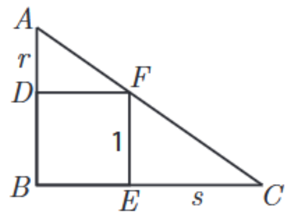
2017A

17. **Answer (D):** The ratio $\frac{PQ}{RS}$ has its greatest value when PQ is as large as possible and RS is as small as possible. Points P, Q, R, and S have coordinates among $(\pm 5, 0)$, $(\pm 4, \pm 3)$, $(\pm 4, \mp 3)$, $(\pm 3, \pm 4)$, $(\pm 3, \mp 4)$, and $(0, \pm 5)$. In order for the distance between two of these points to be irrational, the two points must not form a diameter, and they must not have the same x-coordinate or y-coordinate. If

$R = (a, b)$ and $S = (a', b')$, then $|a - a'| \geq 1$ and $|b - b'| \geq 1$. Because $(3, 4)$ and $(4, 3)$ achieve this, they are as close as two points can be, $\sqrt{2}$ units apart. If $P = (a, b)$ and $Q = (a', b')$, then PQ is maximized when the distance from (a', b') to $(-a, -b)$ is minimized. Because $|a + a'| \geq 1$ and $|b + b'| \geq 1$, the points $(3, -4)$ and $(-4, 3)$ are as far apart as possible, $\sqrt{98}$ units. Therefore the greatest possible ratio is $\frac{\sqrt{98}}{\sqrt{2}} = \sqrt{49} = 7$.

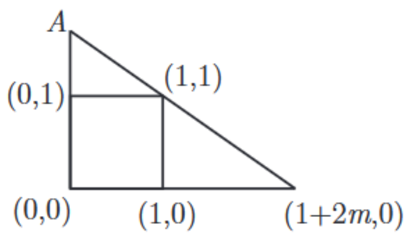
2000

19. **Answer (D):** With out loss of generality, let the side of the square have length 1 unit and let the area of triangle ADF be m. Let $AD = r$ and $EC = s$. Because triangles ADF and FEC are similar, $s/1 = 1/r$. Since $\frac{1}{2}r = m$, the area of triangle FEC is $\frac{1}{2}s = \frac{1}{2r} = \frac{1}{4m}$.



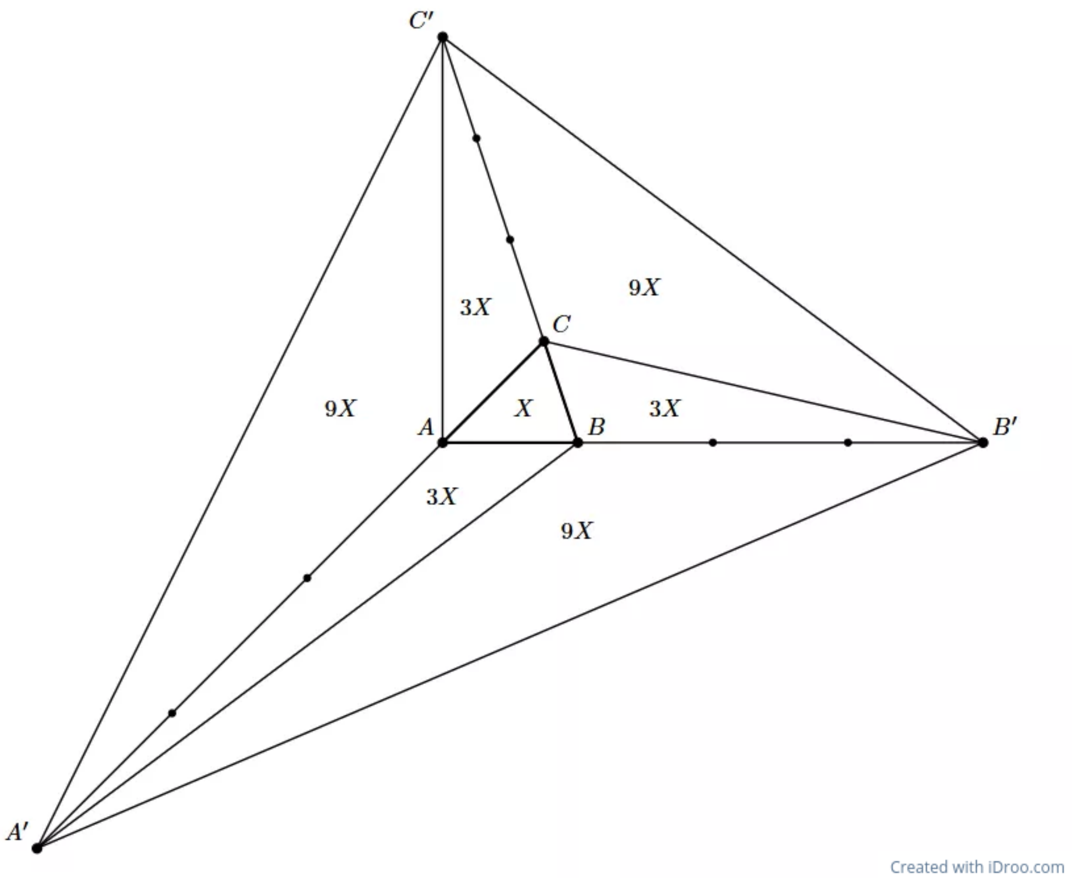
OR

Let $B = (0, 0)$, $E = (1, 0)$, $F = (1, 1)$ and $D = (0, 1)$ be the vertices of the square. Let $C = (1 + 2m, 0)$, and notice that the area of $BEFD$ is 1 and the area of triangle FEC is m. The slope of the line through C and F is $-\frac{1}{2m}$; thus, it intersects the y-axis at $A = \left(0, 1 + \frac{1}{2m}\right)$. The area of triangle ADF is therefore $\frac{1}{4m}$.



2017B

19. **Answer (E):** Draw segments $\overline{CB'}$, $\overline{AC'}$, and $\overline{BA'}$. Let X be the area of $\triangle ABC$. Because $\triangle BB'C$ has a base 3 times as long and the same altitude, its area is $3X$. Similarly, the areas of $\triangle AA'B$ and $\triangle CC'A$ are also $3X$. Furthermore, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$. The areas of $\triangle CC'B'$ and $\triangle BB'A'$ are also $9X$ by the same reasoning. Therefore the area of $\triangle A'B'C'$ is $X + 3(3X) + 3(9X) = 37X$, and the requested ratio is $37 : 1$. Note that nothing in this argument requires $\triangle ABC$ to be equilateral.



2016A

16. **Answer (D):** After reflection about the x-axis, the coordinates of the image are $A'(0,-2)$, $B'(-3,-2)$, and $C'(-3,0)$. The counterclockwise $90°$-rotation around the origin maps this triangle to the triangle with vertices $A''(2,0)$, $B''(2,-3)$, and $C''(0,-3)$. Notice that the final image can be mapped to the original triangle by interchanging the x- and y-coordinates, which corresponds to a reflection about the line $y = x$.

2010B

20. **Answer (D):** It may be assumed that hexagon $ABCDEF$ has side length 1. Let lines BC and FA intersect at G, let H and J be the midpoints of $\overline{AB}$ and $\overline{DE}$, respectively, let K be the center of the second circle, and let that circle be tangent to line BC at L. Equilateral $\triangle ABG$ has side length 1, so the first circle, which is the inscribed circle of $\triangle ABG$, has radius $\frac{\sqrt{3}}{6}$. Let r be the radius of the second circle. Then $\triangle GLK$ is a $30-60-90°$ right triangle with $LK = r$ and $2r = GK = GH + HJ + JK = \frac{\sqrt{3}}{2} + \sqrt{3} + r$. Therefore $r = \frac{3\sqrt{3}}{2} = 9(\frac{\sqrt{3}}{6})$. The ratio of the radii of the two circles is 9, and the ratio of their areas is $9^2 = 81$.