18

SEQUENCE AND SERIES

2016B

16. **Answer (E):** Let r be the common ratio of the geometric series; then

$$S = \frac{1}{r} + 1 + r + r^2 + \cdots = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2}.$$

Because $S > 0$, the smallest value of S occurs when the value of $r - r^2$ is maximized. The graph of $f(r) = r - r^2$ is a downward-opening parabola with vertex $(\frac{1}{2}, \frac{1}{4})$, so the smallest possible value of S is $\frac{1}{(\frac{1}{4})} = 4$. The optimal series is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

2018B

16. **Answer (E):** Let n be an integer. Because $n^3 - n = (n-1)n(n+1)$, it follows that $n^3 - n$ has at least one prime factor of 2 and one prime factor of 3 and therefore is divisible by 6. Thus $n^3 \equiv n \pmod 6$. Then

$$a_1^3 + a_2^3 + \cdots + a_{2018}^3 \equiv a_1 + a_2 + \cdots + a_{2018} \equiv 2018^{2018} \pmod 6.$$

Because $2018 \equiv 2 \pmod 6$, the powers of 2018 modulo 6 are alternately $2, 4, 2, 4, \ldots$, so $2018^{2018} \equiv 4 \pmod 6$. Therefore the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6 is 4.

2004A

18. **(A)** The terms of the arithmetic progression are 9, $9+d$, and $9+2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

$$(11+d)^2 = 9(29+2d) \quad \text{so} \quad d^2 + 4d - 140 = 0.$$

Thus $d = 10$ or $d = -14$. The corresponding geometric progressions are 9, 21, 49 and 9, -3, 1, so the smallest possible value for the third term of the geometric progression is 1.

2006B

18. **(E)** Note that the first several terms of the sequence are:

$$2,\ 3,\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ 2,\ 3,\ \ldots,$$

so the sequence consists of a repeating cycle of 6 terms. Since $2006 = 334 \cdot 6 + 2$, we have $a_{2006} = a_2 = 3$.

2002B

19. **(C)** Let $d = a_2 - a_1$. Then $a_{k+100} = a_k + 100d$, and

$$\begin{aligned} a_{101} + a_{102} + \cdots + a_{200} &= (a_1 + 100d) + (a_2 + 100d) + \ldots + (a_{100} + 100d) \\ &= a_1 + a_2 + \ldots + a_{100} + 10{,}000d. \end{aligned}$$

Thus $200 = 100 + 10{,}000d$ and $d = \frac{100}{10{,}000} = 0.01$.

2004B 19. **(C)** Let a_k be the k^{th} term of the sequence. For $k \geq 3$,

$$a_{k+1} = a_{k-2} + a_{k-1} - a_k, \quad \text{so} \quad a_{k+1} - a_{k-1} = -(a_k - a_{k-2}).$$

Because the sequence begins

$$2001, 2002, 2003, 2000, 2005, 1998, \ldots,$$

it follows that the odd-numbered terms and the even-numbered terms each form arithmetic progressions with common differences of 2 and -2, respectively. The 2004^{th} term of the original sequence is the 1002^{nd} term of the sequence 2002, 2000, 1998,. . ., and that term is $2002 + 1001(-2) = 0$.

2006A 19. **(C)** Let $n - d$, n, and $n + d$ be the angles in the triangle. Then

$$180 = n - d + n + n + d = 3n, \quad \text{so} \quad n = 60.$$

Because the sum of the degree measures of two angles of a triangle is less than 180, we have

$$180 > n + (n + d) = 120 + d, \quad \text{which implies that } 0 < d < 60.$$

There are 59 triangles with this property.

2017A 20. **Answer (D):** Note that $S(n+1) = S(n)+1$ unless the numeral for n ends with a 9. Moreover, if the numeral for n ends with exactly k 9s, then $S(n+1) = S(n)+1-9k$. Thus the possible values of $S(n+1)$ when $S(n) = 1274$ are all of the form $1275-9k$, where $k \in \{0,1,2,3,\ldots,141\}$. Of the choices, only 1239 can be formed in this manner, and $S(n+1)$ will equal 1239 if, for example, n consists of 4 consecutive 9s preceded by 1238 1s.

OR

The value of a positive integer is congruent to the sum of its digits modulo 9. Therefore $n \equiv S(n) = 1274 \equiv 5 \pmod 9$, so $S(n+1) \equiv n+1 \equiv 6 \pmod 9$. Of the given choices, only 1239 meets this requirement.