19

QUADRATICS

2005B

16. **(D)** Let r_1 and r_2 be the roots of $x^2+px+m=0$. Since the roots of $x^2+mx+n=0$ are $2r_1$ and $2r_2$, we have the following relationships:

$$m = r_1r_2, \quad n = 4r_1r_2, \quad p = -(r_1+r_2), \quad \text{and} \quad m = -2(r_1+r_2).$$

So

$$n = 4m, \quad p = \frac{1}{2}m, \quad \text{and} \quad \frac{n}{p} = \frac{4m}{\frac{1}{2}m} = 8.$$

OR

The roots of

$$\left(\frac{x}{2}\right)^2 + p\left(\frac{x}{2}\right) + m = 0$$

are twice those of $x^2+px+m=0$. Since the first equation is equivalent to $x^2+2px+4m=0$, we have

$$m = 2p \quad \text{and} \quad n = 4m, \quad \text{so} \quad \frac{n}{p} = 8.$$

2015A

16. **Answer (B):** Expanding the binomials and subtracting the equations yields $x^2 - y^2 = 3(x-y)$. Because $x - y \neq 0$, it follows that $x + y = 3$. Adding the equations gives $x^2 + y^2 = 5(x+y) = 5 \cdot 3 = 15$.

Note: The two solutions are $(x, y) = (\frac{3}{2} + \frac{\sqrt{21}}{2}, \frac{3}{2} - \frac{\sqrt{21}}{2})$ and $(\frac{3}{2} - \frac{\sqrt{21}}{2}, \frac{3}{2} + \frac{\sqrt{21}}{2})$.

2003A

18. (B) Let $a = 2003/2004$. The given equation is equivalent to

$$ax^2 + x + 1 = 0.$$

If the roots of this equation are denoted r and s, then

$$rs = \frac{1}{a} \quad \text{and} \quad r + s = -\frac{1}{a},$$

so

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = -1.$$

OR

If x is replaced by $1/y$, then the roots of the resulting equation are the reciprocals of the roots of the original equation. The new equation is

$$\frac{2003}{2004y} + 1 + y = 0 \quad \text{which is equivalent to} \quad y^2 + y + \frac{2003}{2004} = 0.$$

The sum of the roots of this equation is the opposite of the y-coefficient, which is -1.

2011B

19. **Answer (A):** The right side of the equation is defined only when $|x| \geq 4$. If $x \geq 4$, the equation is equivalent to $5x + 8 = x^2 - 16$, and the only solution with $x \geq 4$ is $x = 8$. If $x \leq -4$, the equation is equivalent to $8 - 5x = x^2 - 16$, and the only solution with $x \leq -4$ is $x = -8$. The product of the solutions is $-8 \cdot 8 = -64$.

2013B

19. **Answer (D):** Let the common difference in the arithmetic sequence be d, so that $a = b + d$ and $c = b - d$. Because the quadratic has exactly one root, $b^2 - 4ac = 0$. Substitution gives $b^2 = 4(b + d)(b - d)$, and therefore $3b^2 = 4d^2$. Because $b \geq 0$ and $d \geq 0$, it follows that $\sqrt{3}b = 2d$. Thus the real root is

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b}{2a} = \frac{-b}{2(b+d)} = \frac{-b}{2\left(b + \frac{\sqrt{3}}{2}b\right)} = -2 + \sqrt{3}.$$

Note that the quadratic equation $x^2 + (4 - 2\sqrt{3})x + 7 - 4\sqrt{3}$ satisfies the given conditions.

2007A

20. **Answer (D):** Squaring each side of the equation $4 = a + a^{-1}$ gives

$$16 = a^2 + 2a \cdot a^{-1} + \left(a^{-1}\right)^2 = a^2 + 2 + a^{-2}, \quad \text{so} \quad 14 = a^2 + a^{-2}.$$

Squaring again gives

$$196 = a^4 + 2a^2 \cdot a^{-2} + \left(a^{-2}\right)^2 = a^4 + 2 + a^{-4}, \quad \text{so} \quad 194 = a^4 + a^{-4}.$$

2014B

20. **Answer (C):** Note that $x^4 - 51x^2 + 50 = (x^2 - 50)(x^2 - 1)$, so the roots of the polynomial are ± 1 and $\pm\sqrt{50}$. Arranged from least to greatest, these roots are approximately $-7.1, -1, 1, 7.1$. The polynomial takes negative values on the intervals $(-7.1, -1)$ and $(1, 7.1)$, which include 12 integers: -7, -6, -5, -4, -3, -2, 2, 3, 4, 5, 6, 7.