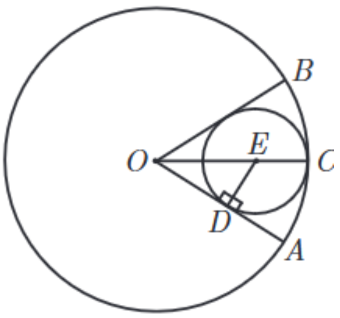
3

2D WORD PROBLEMS

2008A

16. **Answer (B):** Let r and R be the radii of the smaller and larger circles, respectively. Let E be the center of the smaller circle, let $\overline{OC}$ be the radius of the larger circle that contains E, and let D be the point of tangency of the smaller circle to $\overline{OA}$. Then $OE = R - r$, and because $\triangle EDO$ is a $30-60-90^\circ$ triangle, $OE = 2DE = 2r$. Thus $2r = R - r$, so $\frac{r}{R} = \frac{1}{3}$. The ratio of the areas is $(\frac{1}{3})^2 = \frac{1}{9}$.

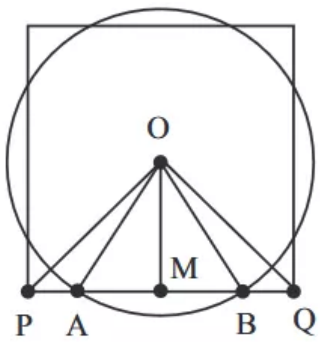


2010B

16. **Answer (B):** Let O be the common center of the circle and the square. Let M be the midpoint of a side of the square and P and Q be the vertices of the square on the side containing M. Since

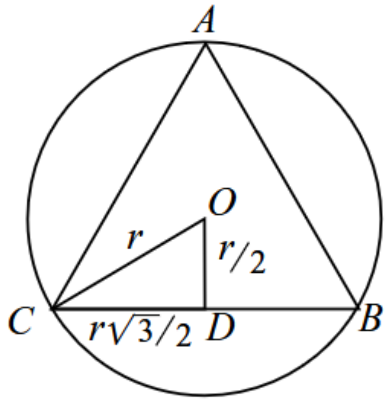
$$OM^2 = \left(\frac{1}{2}\right)^2 < \left(\frac{\sqrt{3}}{3}\right)^2 < \left(\frac{\sqrt{2}}{2}\right)^2 = OP^2 = OQ^2,$$

the midpoint of each side is inside the circle and the vertices of the square are outside the circle. Therefore the circle intersects the square in two points along each side.



Let A and B be the intersection points of the circle with $\overline{PQ}$. Then M is also the midpoint of $\overline{AB}$ and $\triangle OMA$ is a right triangle. By the Pythagorean Theorem $AM = \frac{1}{2\sqrt{3}}$, so $\triangle OMA$ is a $30-60-90^\circ$ right triangle. Then $\angle AOB = 60^\circ$, and the area of the sector corresponding to $\angle AOB$ is $\frac{1}{6} \cdot \pi \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\pi}{18}$. The area of $\triangle AOB$ is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{12}$. The area outside the square but inside the circle is $4 \cdot \left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) = \frac{2\pi}{9} - \frac{\sqrt{3}}{3}$.

2003A 17. (B) Let the triangle have vertices A, B, and C, let O be the center of the circle, and let D be the midpoint of $\overline{BC}$. Triangle COD is a 30–60–90 degree triangle. If r is the radius of the circle, then the sides of $\triangle COD$ are r, $r/2$, and $r\sqrt{3}/2$. The perimeter of $\triangle ABC$ is $6\left(\frac{r\sqrt{3}}{2}\right) = 3r\sqrt{3}$, and the area of the circle is πr^2. Thus $3r\sqrt{3} = \pi r^2$, and $r = (3\sqrt{3})/\pi$.

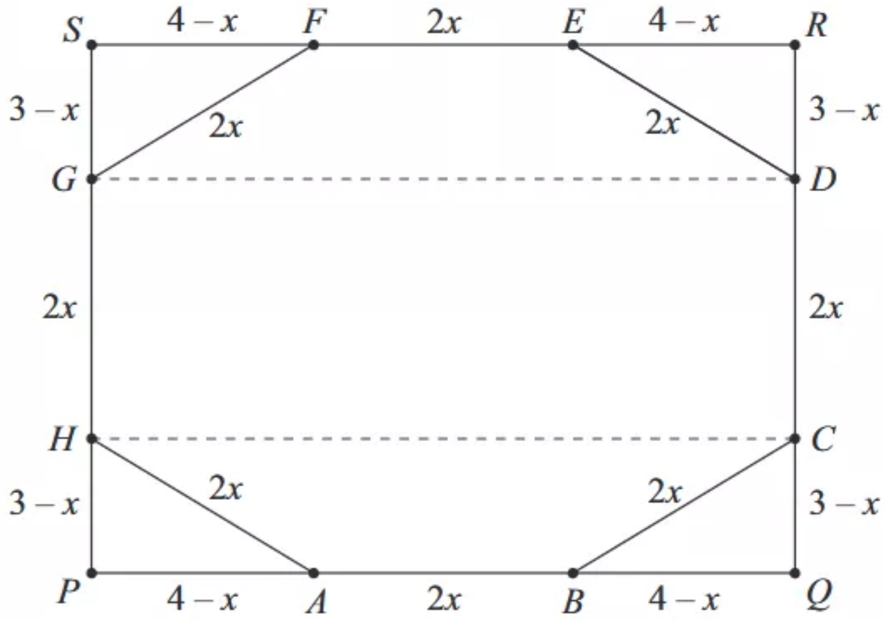


2007B 17. **Answer (D):** Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

2018B

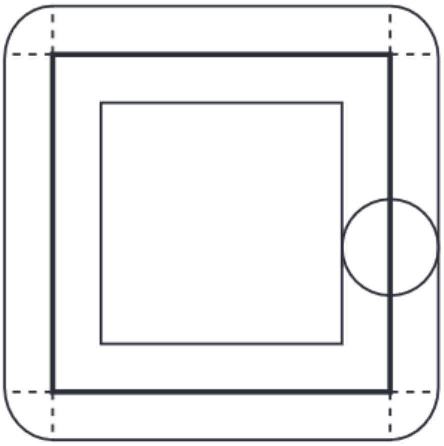
17. **Answer (B):** Because $AP < 4 = \frac{1}{2}PQ$, it follows that A is closer to P than it is to Q and that A is between points P and B. Because $AP = BQ$, $AH = BC$, and angles APH and BQC are right angles, $\triangle APH \cong \triangle BQC$. Thus $PH = QC$, and $PQCH$ is a rectangle. Because $CD = HG$, it follows that $HCDG$ is also a rectangle. Thus $GDRS$ is a rectangle and $DR = GS$, and it follows that $\triangle ERD \cong \triangle FSG$. Therefore segment $\overline{EF}$ is centered in $\overline{RS}$ just as congruent segment $\overline{AB}$ is centered in $\overline{PQ}$. Therefore $\triangle ERD \cong \triangle BQC$, and $\overline{CD}$ is also centered in $\overline{QR}$. Let $2x$ be the side length $AB = BC =$

$CD = DE = EF = FG = GH = HA$ of the regular octagon; then $AP = BQ = 4-x$ and $QC = RD = 3-x$. Applying the Pythagorean Theorem to $\triangle BQC$ yields $(4-x)^2+(3-x)^2 = (2x)^2$, which simplifies to $2x^2 + 14x - 25 = 0$. Thus $x = \frac{1}{2} \cdot (-7 \pm 3\sqrt{11})$, and because $x > 0$, it follows that $2x = -7 + 3\sqrt{11}$. Hence $k + m + n = -7 + 3 + 11 = 7$.



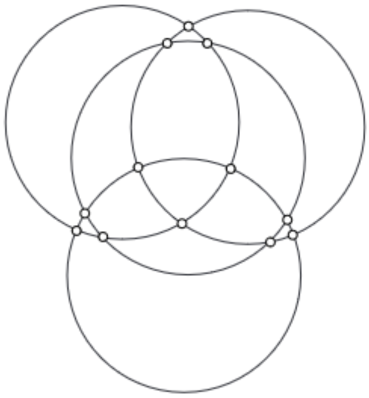
2000

18. **Answer (C):** At any point on Charlyn's walk, she can see all the points inside a circle of radius 1 km. The portion of the viewable region inside the square consists of the interior of the square except for a smaller square with side length 3 km. This portion of the viewable region has area $(25 - 9)$ km^2. The portion of the viewable region outside the square consists of four rectangles, each 5 km by 1 km, and four quarter-circles, each with a radius of 1 km. This portion of the viewable region has area $4(5 + \frac{\pi}{4}) = (20 + \pi)$ km^2. The area of the entire viewable region is $36 + \pi \approx 30$ km^2.



2002B

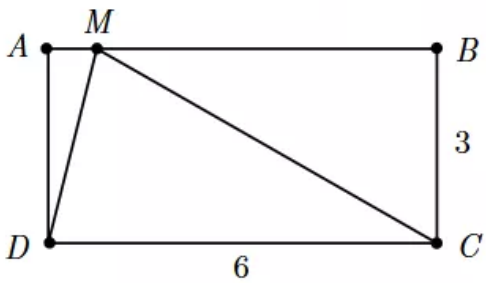
18. **(D)** Each pair of circles has at most two intersection points. There are $\binom{4}{2} = 6$ pairs of circles, so there are at most $6 \times 2 = 12$ points of intersection. The following configuration shows that 12 points of intersection are indeed possible:



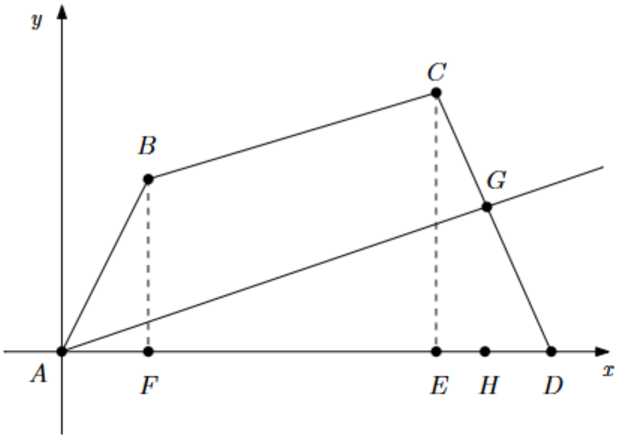
2011B

18. **Answer (E):** Sides $\overline{AB}$ and $\overline{CD}$ are parallel, so $\angle CDM = \angle AMD$. Because $\angle AMD = \angle CMD$, it follows that $\triangle CMD$ is isosceles and $CD = CM = 6$.

Therefore $\triangle MCB$ is a $30-60-90°$ right triangle with $\angle BMC = 30°$. Finally, $2 \cdot \angle AMD + 30° = \angle AMD + \angle CMD + 30° = 180°$, so $\angle AMD = 75°$.



2013A 18. **Answer (B):** Let line AG be the required line, with G on $\overline{CD}$. Divide $ABCD$ into triangle ABF, trapezoid $BCEF$, and triangle CDE, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x-4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p+q+r+s = 58$.

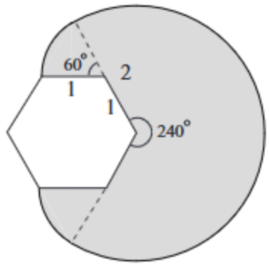


2014A 18. **Answer (B):** Let the square have vertices A, B, C, D in counterclockwise order. Without loss of generality assume that $A = (0,0)$ and $B = (x,1)$ for some $x > 0$. Because D is the image of B under a 90° counterclockwise rotation about A, the coordinates of D are $(-1,x)$, so $x = 4$. Therefore the area of the square is $(AB)^2 = 4^2 + 1^2 = 17$. Note that $C = (3,5)$, and $ABCD$ is indeed a square.

2002A

19. **(E)** Spot can go anywhere in a $240°$ sector of radius two yards and can cover a $60°$ sector of radius one yard around each of the adjoining corners. The total area is

$$\pi(2)^2 \cdot \frac{240}{360} + 2\left(\pi(1)^2 \cdot \frac{60}{360}\right) = 3\pi.$$



2008A

19. **Answer (C):** Let P' and S' denote the positions of P and S, respectively, after the rotation about R, and let P'' denote the final position of P. In the rotation that moves P to position P', the point P rotates $90°$ on a circle with center R and radius $PR = \sqrt{2^2+6^2} = 2\sqrt{10}$. The length of the arc traced by P is $(1/4)\left(2\pi \cdot 2\sqrt{10}\right) = \pi\sqrt{10}$. Next, P' rotates to P'' through a $90°$ arc on a circle with center S' and radius $S'P' = 6$. The length of this arc is $\frac{1}{4}(2\pi \cdot 6) = 3\pi$. The total distance traveled by P is

$$\pi\sqrt{10} + 3\pi = \left(3+\sqrt{10}\right)\pi.$$

2010A

19. **Answer (E):** Triangles ABC, CDE and EFA are congruent, so $\triangle ACE$ is equilateral. Let X be the intersection of the lines AB and EF and define Y and Z similarly as shown in the figure. Because $ABCDEF$ is equiangular, it follows that $\angle XAF = \angle AFX = 60°$. Thus $\triangle XAF$ is equilateral. Let H be the midpoint of $\overline{XF}$. By the Pythagorean Theorem,

$$AE^2 = AH^2 + HE^2 = (\frac{\sqrt{3}}{2}r)^2 + (\frac{r}{2} + 1)^2 = r^2 + r + 1$$

Thus, the area of $\triangle ACE$ is

$$\frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1).$$

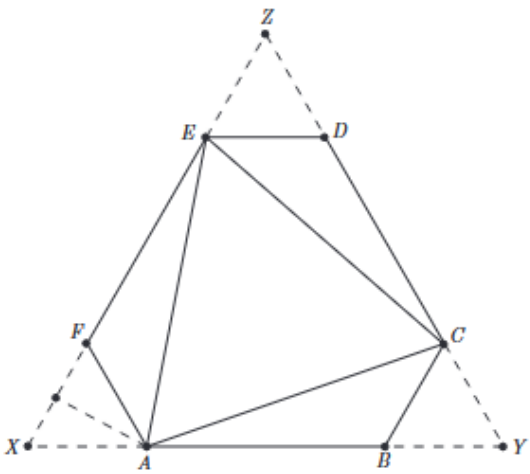
The area of hexagon $ABCDEF$ is equal to

$$[XYZ] - [XAF] - [YCB] - [ZED] = \frac{\sqrt{3}}{4}\left((2r+1)^2 - 3r^2\right) = \frac{\sqrt{3}}{4}(r^2 + 4r + 1)$$

Because $[ACE] = \frac{7}{10}[ABCDEF]$, it follows that

$$r^2 + r + 1 = \frac{7}{10}(r^2 + 4r + 1)$$

from which $r^2 - 6r + 1 = 0$ and $r = 3 \pm 2\sqrt{2}$. The sum of all possible values of r is 6.



2012B

19. **Answer (D):** Let the length of the lunch break be m minutes. Then the three painters each worked $480-m$ minutes on Monday, the two helpers worked $372-m$ minutes on Tuesday, and Paula worked $672-m$ minutes on Wednesday. If Paula paints $p\%$ of the house per minute and her helpers paint a total of $h\%$ of the house per minute, then

$$\begin{aligned}(p+h)(480-m) &= 50,\\ h(372-m) &= 24, \text{and}\\ p(672-m) &= 26.\end{aligned}$$

Adding the last two equations gives $672p+372h-mp-mh=50$, and subtracting this equation from the first one gives $108h-192p=0$, so $h=\frac{16p}{9}$. Substitution into the first equation then leads to the system

$$\begin{aligned}\frac{25p}{9}(480-m) &= 50,\\ p(672-m) &= 26.\end{aligned}$$

The solution of this system is $p=\frac{1}{24}$ and $m=48$. Note that $h=\frac{2}{27}$.

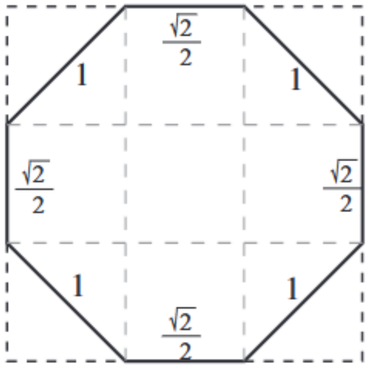
2001

20. **(B)** Let x represent the length of each side of the octagon, which is also the length of the hypotenuse of each of the right triangles. Each leg of the right triangles has length $x\sqrt{2}/2$, so

$$2\cdot\frac{x\sqrt{2}}{2}+x=2000, \text{ and } x=\frac{2000}{\sqrt{2}+1}=2000(\sqrt{2}-1).$$

2005A 20. **(A)** The octagon can be partitioned into five squares and four half squares, each with side length $\sqrt{2}/2$, so its area is

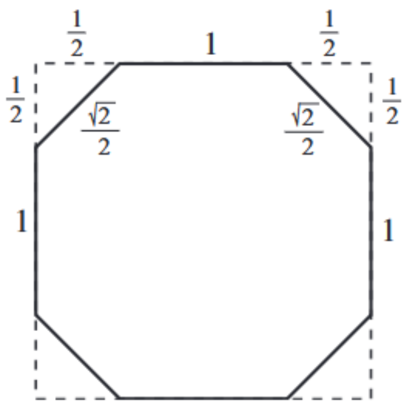
$$\left(5+4\cdot\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^2=\frac{7}{2}.$$



OR

The octagon can be obtained by removing four isosceles right triangles with legs of length 1/2 from a square with sides of length 2. Thus its area is

$$2^2-4\cdot\frac{1}{2}\left(\frac{1}{2}\right)^2=\frac{7}{2}.$$



2006B

20. **(E)** The slope of line AB is $(178-(-22))/(2006-6) = 1/10$. Since the line AD is perpendicular to the line AB, its slope is -10. This implies that

$$-10 = \frac{y-(-22)}{8-6}, \quad \text{so} \quad y = -10(2) - 22 = -42, \quad \text{and} \quad D = (8, -42).$$

As a consequence,

$$AB = \sqrt{2000^2 + 200^2} = 200\sqrt{101} \quad \text{and} \quad AD = \sqrt{2^2 + 20^2} = 2\sqrt{101}.$$

Thus

$$\text{Area}(ABCD) = AB \cdot AD = 400 \cdot 101 = 40{,}400.$$

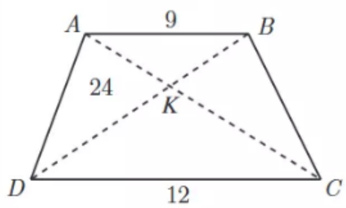
2008A

20. **Answer (D):** Note that $\triangle ABK$ is similar to $\triangle CDK$. Because $\triangle AKD$ and $\triangle KCD$ have collinear bases and share a vertex D,

$$\frac{\text{Area}(\triangle KCD)}{\text{Area}(\triangle AKD)} = \frac{KC}{AK} = \frac{CD}{AB} = \frac{4}{3},$$

so $\triangle KCD$ has area 32.

By a similar argument, $\triangle KAB$ has area 18. Finally, $\triangle BKC$ has the same area as $\triangle AKD$ since they are in the same proportion to each of the other two triangles. The total area is $24 + 32 + 18 + 24 = 98$.



OR

Let h denote the height of the trapezoid. Then

$$24 + \text{Area}(\triangle AKB) = \frac{9h}{2}.$$

Because $\triangle CKD$ is similar to $\triangle AKB$ with similarity ratio $\frac{12}{9} = \frac{4}{3}$,

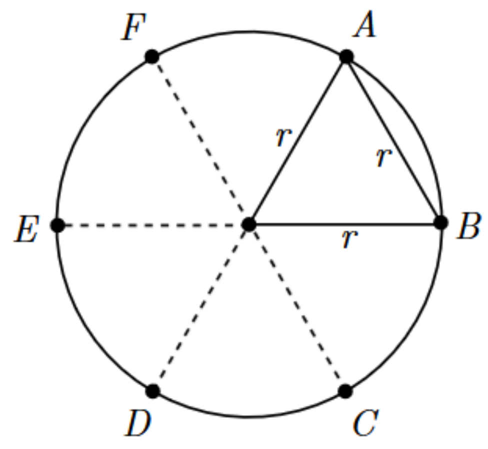
$$\text{Area}(\triangle CKD) = \frac{16}{9}\text{Area}(\triangle AKB), \quad \text{so} \quad 24 + \frac{16}{9}\text{Area}(\triangle AKB) = \frac{12h}{2}.$$

Solving the two equations simultaneously yields $h = \frac{28}{3}$. This implies that the area of the trapezoid is

$$\frac{1}{2} \cdot \frac{28}{3}(9 + 12) = 98.$$

2011A

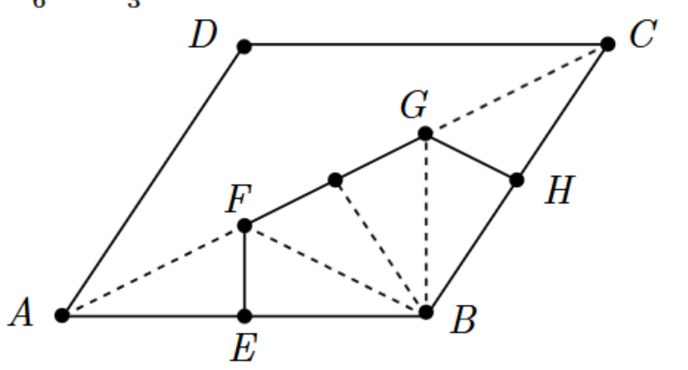
20. **Answer (D):** Let point A be the first point chosen, and let point B be the opposite endpoint of the corresponding chord. Drawing a radius to each endpoint of this chord of length r results in an equilateral triangle. Hence a chord of length r subtends an arc $\frac{1}{6}$ the circumference of the circle. Let diameter $\overline{FC}$ be parallel to $\overline{AB}$, and divide the circle into six equal portions as shown. The second point chosen will result in a chord that intersects $\overline{AB}$ if and only if the point is chosen from minor $\overset{\frown}{FB}$. Hence the probability is $\frac{1}{3}$.



2011B

20. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from the midpoint of $\overline{BC}$. Then the desired region R is the pentagon $BEFGH$.

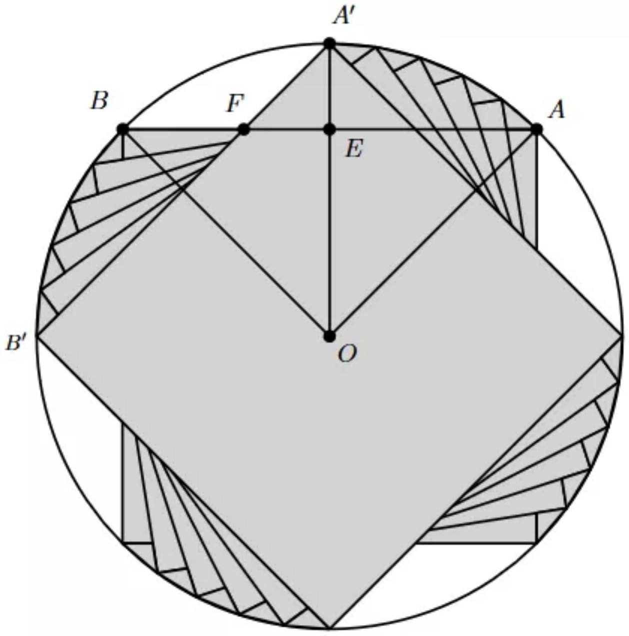
Note that $\triangle AFE$ is a $30-60-90°$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120° - \angle FBE - \angle GBH = 60°$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



2013A

20. Answer (C): Let O be the center of unit square $ABCD$, let A and B be rotated to points A' and B', and let $\overline{OA'}$ and $\overline{A'B'}$ intersect $\overline{AB}$ at E and F, respectively. Then one quarter of the region swept out by the interior of the square consists of the $45°$ sector AOA' with radius $\frac{\sqrt{2}}{2}$, isosceles right triangle OEB with leg length $\frac{1}{2}$, and isosceles right triangle $A'EF$ with leg length $\frac{\sqrt{2}-1}{2}$. Thus the area of the region is

$$4\left(\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}-1}{2}\right)^2\right)=2-\sqrt{2}+\frac{\pi}{4}.$$



2016B

20. **Answer (C):** The scale factor for this transformation is $\frac{3}{2}$. The center of the dilation, D, must lie along ray $A'A$ (with A between A' and D), and its distance from A must be $\frac{2}{3}$ of its distance from A'. Because A is 3 units to the left of and 4 units below A', the center of the dilation must be 6 units to the left of and 8 units below A, placing it at $D(-4,-6)$. The origin is $\sqrt{(-4)^2+(-6)^2}=2\sqrt{13}$ units from D, so the dilation must move it half that far, or $\sqrt{13}$ units. Alternatively, note that the origin is 4 units to the right of and 6 units above D, so its image must be 6 units to the right of and 9 units above D; therefore it is located at $(2,3)$, a distance $\sqrt{2^2+3^2}=\sqrt{13}$ from the origin.