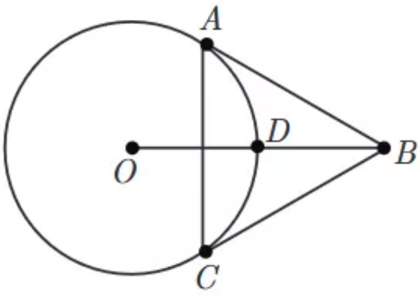
4

TRIANGLES

2009B

16. **Answer (B):** Let the radius of the circle be r. Because $\triangle BCO$ is a right triangle with a 30° angle at B, the hypotenuse $\overline{BO}$ is twice as long as $\overline{OC}$, so $BO = 2r$. It follows that $BD = 2r - r = r$, and

$$\frac{BD}{BO} = \frac{r}{2r} = \frac{1}{2}.$$

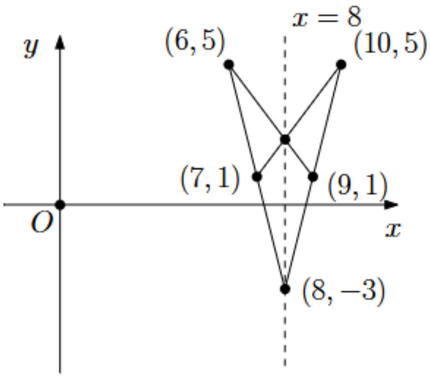


2010A

16. **Answer (B):** By the Angle Bisector Theorem, $8 \cdot BA = 3 \cdot BC$. Thus BA must be a multiple of 3. If $BA = 3$, the triangle is degenerate. If $BA = 6$, then $BC = 16$, and the perimeter is $6 + 16 + 11 = 33$.

2013A

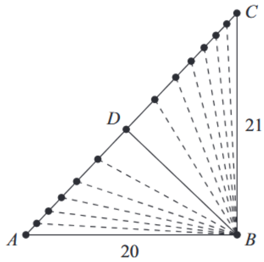
16. **Answer (E):** The reflected triangle has vertices $(7, 1)$, $(8, -3)$, and $(10, 5)$. The point $(9, 1)$ is on the line segment from $(10, 5)$ to $(8, -3)$. The line segment from $(6, 5)$ to $(9, 1)$ contains the point $(8, \frac{7}{3})$, which must be on both triangles, and by symmetry the point $(7, 1)$ is on the line segment from $(6, 5)$ to $(8, -3)$. Therefore the union of the two triangles is also the union of two congruent triangles with disjoint interiors, each having the line segment from $(8, -3)$ to $(8, \frac{7}{3})$ as a base. The altitude of one of the two triangles is the distance from the line $x = 8$ to the point $(10, 5)$, which is 2. Hence the union of the triangles has area $2 \cdot (\frac{1}{2} \cdot 2 \cdot (\frac{7}{3} + 3)) = \frac{32}{3}$.



2016A

2018A

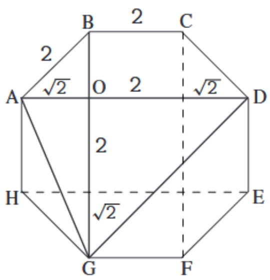
16. **Answer (D):** The area of $\triangle ABC$ is 210. Let D be the foot of the altitude from B to $\overline{AC}$. By the Pythagorean Theorem, $AC = \sqrt{20^2 + 21^2} = 29$, so $210 = \frac{1}{2} \cdot 29 \cdot BD$, and $BD = 14\frac{14}{29}$. Two segments of every length from 15 through 19 can be constructed from B to $\overline{AC}$. In addition to these 10 segments and the 2 legs, there is a segment of length 20 from B to a point on $\overline{AC}$ near C, for a total of 13 segments with integer length.



2002B

17. **(C)** Construct the right triangle $\triangle AOB$ as shown in the figure. Since $AB = 2$, we have $AO = \sqrt{2}$ and $AD = 2 + 2\sqrt{2}$. Similarly, we have $OG = 2 + \sqrt{2}$, so

$$\text{Area}(\triangle ADG) = \frac{1}{2}(2+2\sqrt{2})(2+\sqrt{2}) = (1+\sqrt{2})(2+\sqrt{2}) = 4 + 3\sqrt{2}.$$

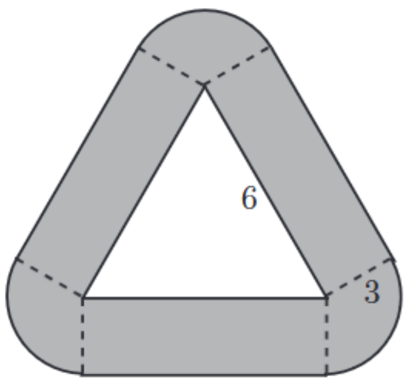


2008A

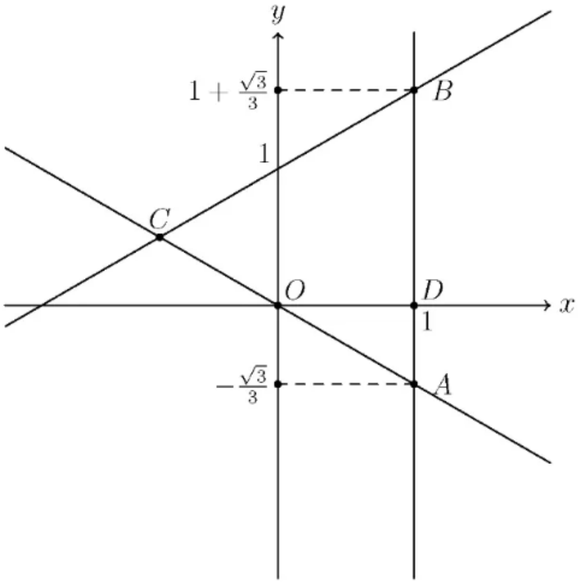
17. **Answer (B):** The region consists of three rectangles with length 6 and width 3 together with three 120° sectors of circles with radius 3.

The combined area of the three 120° sectors is the same as the area of a circle with radius 3, so the area of the region is

$$3 \cdot 6 \cdot 3 + \pi \cdot 3^2 = 54 + 9\pi.$$



2015A 17. **Answer (D):** Label the vertices of the equilateral triangle A, B, and C so that A is on the line $x = 1$ and B is on both lines $x = 1$ and $y = 1 + \frac{\sqrt{3}}{3}x$. Then $B = (1, 1 + \frac{\sqrt{3}}{3})$. Let O be the origin and $D = (1, 0)$. Because $\triangle ABC$ is equilateral, $\angle CAB = 60^\circ$, and $\triangle OAD$ is a $30-60-90^\circ$ triangle. Because $OD = 1$, $AD = \frac{\sqrt{3}}{3}$ and $AB = AD + DB = \frac{\sqrt{3}}{3} + (1 + \frac{\sqrt{3}}{3}) = 1 + \frac{2\sqrt{3}}{3}$. The perimeter of $\triangle ABC$ is $3 \cdot AB = 3 + 2\sqrt{3}$. Indeed, $\triangle ABC$ is equilateral with $C = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$.



2008A 18. **Answer (B):** Let x be the length of the hypotenuse, and let y and z be the lengths of the legs. The given conditions imply that

$$y^2 + z^2 = x^2, \quad y + z = 32 - x, \quad \text{and} \quad yz = 40.$$

Thus

$$(32 - x)^2 = (y + z)^2 = y^2 + z^2 + 2yz = x^2 + 80,$$

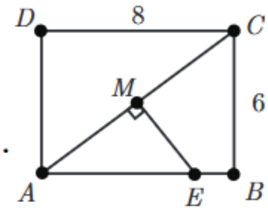
from which $1024 - 64x = 80$, and $x = \frac{59}{4}$.

Note: Solving the system of equations yields leg lengths of

$$\frac{1}{8}(69 + \sqrt{2201}) \quad \text{and} \quad \frac{1}{8}(69 - \sqrt{2201}),$$

so a triangle satisfying the given conditions does in fact exist.

2009B



18. **Answer (D):** By the Pythagorean Theorem, $AC = 10$, so $AM = 5$. Triangles AME and ABC are similar, so $\frac{ME}{AM} = \frac{6}{8}$ and $ME = \frac{15}{4}$. The area of $\triangle AME$ is $\frac{1}{2} \cdot 5 \cdot \frac{15}{4} = \frac{75}{8}$.

OR

As above, $AM = 5$ and $\triangle AME$ and $\triangle ABC$ are similar with similarity ratio 5:8. Therefore

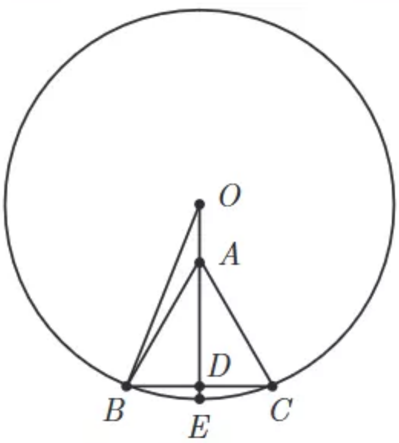
$$Area(\triangle AME) = \left(\frac{5}{8}\right)^2 \cdot Area(\triangle ABC) = \frac{5^2}{8^2} \cdot \frac{8 \cdot 6}{2} = \frac{75}{8}.$$

2010B

19. **Answer (B):** The radius of circle O is $\sqrt{156} > 4\sqrt{3} = OA$, so A is inside the circle. Let s be the side length of $\triangle ABC$, let D be the foot of the altitude from A, and let $\overline{OE}$ be the radius through A. This radius is perpendicular to $\overline{BC}$ and contains D, so $OD = \sqrt{OB^2 - BD^2} = \sqrt{156 - \frac{1}{4}s^2}$. If A is on $\overline{DE}$, then $\angle BAC > \angle BEC > 90°$, an impossibility. Therefore A lies on $\overline{OD}$, and $OA = OD - AD$, that is,

$$4\sqrt{3} = \sqrt{156 - \frac{1}{4}s^2} - \frac{\sqrt{3}}{2}s.$$

Rearranging terms and squaring both sides leads to the quadratic equation $s^2 + 12s - 108 = 0$, and the positive solution is $s = 6$.



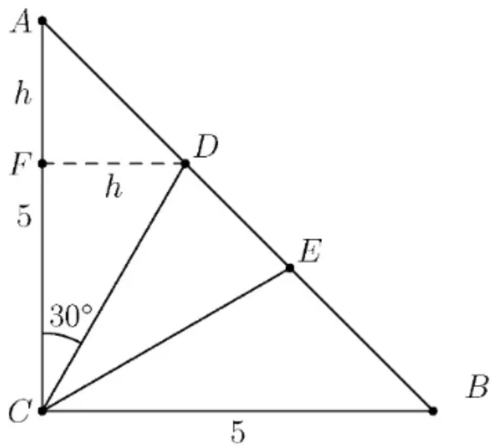
2015A

19. **Answer (D):** Because the area is 12.5, it follows that $AC = BC = 5$. Label D and E so that D is closer to A than to B. Let F be the foot of the perpendicular to $\overline{AC}$ passing through D. Let $h = FD$. Then $AF = h$ because $\triangle ADF$ is an isosceles right triangle, and $CF = h\sqrt{3}$ because $\triangle CDF$ is a $30-60-90^\circ$ triangle. So $h + h\sqrt{3} = AC = 5$ and

$$h = \frac{5}{1+\sqrt{3}} = \frac{5\sqrt{3}-5}{2}.$$

Thus the area of $\triangle CDE$ is

$$\frac{25}{2} - 2 \cdot \frac{1}{2} \cdot 5 \cdot \frac{5\sqrt{3}-5}{2} = \frac{50-25\sqrt{3}}{2}.$$



2015B

19. **Answer (C):** Let O be the center of the circle on which X, Y, Z, and W lie. Then O lies on the perpendicular bisectors of segments $\overline{XY}$ and $\overline{ZW}$, and $OX = OW$. Note that segments $\overline{XY}$ and $\overline{AB}$ have the same perpendicular bisector and segments $\overline{ZW}$ and $\overline{AC}$ have the same perpendicular bisector, from which it follows that O lies on the perpendicular bisectors of segments $\overline{AB}$ and $\overline{AC}$; that is, O is the circumcenter of $\triangle ABC$. Because $\angle C = 90^\circ$, O is the midpoint of hypotenuse $\overline{AB}$. Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$ and $12^2 + 6^2 = OX^2 = OW^2 = b^2 + (a + 2b)^2$. Solving these two equations simultaneously gives $a = b = 3\sqrt{2}$. Thus the perimeter of $\triangle ABC$ is $12+2a+2b = 12 + 12\sqrt{2}$.