9

COMBINATIONS

2003B

16. (E) Let m denote the number of main courses needed to meet the requirement. Then the number of dinners available is $3 \cdot m \cdot 2m = 6m^2$. Thus m^2 must be at least $365/6 \approx 61$. Since $7^2 = 49 < 61 < 64 = 8^2$, 8 main courses is enough, but 7 is not.

2017B

16. **Answer (A):** It will be easier to count the complementary set. There are 9 one-digit numerals that do not contain the digit 0, $9 \cdot 9 = 81$ two-digit numerals that do not contain the digit 0, $9 \cdot 9 \cdot 9 = 729$ three-digit numerals that do not contain the digit 0, and $1 \cdot 9 \cdot 9 \cdot 9 = 729$ four-digit numerals starting with 1 that do not contain the digit 0, a total of 1548. All four-digit numerals between 2000 and 2017, inclusive, contain the digit 0. Therefore $2017 - 1548 = 469$ numerals in the required range do contain the digit 0.

2005B

18. **(D)** The last seven digits of the phone number use seven of the eight digits $\{2, 3, 4, 5, 6, 7, 8, 9\}$, so all but one of these digits is used. The unused digit can be chosen in eight ways. The remaining seven digits are then placed in increasing order to obtain a possible phone number. Thus there are 8 possible phone numbers.

2006A

18. **(C)** Since the two letters have to be next to each other, think of them as forming a two-letter word w. So each license plate consists of 4 digits and w. For each digit there are 10 choices. There are $26 \cdot 26$ choices for the letters of w, and there are 5 choices for the position of w. So the total number of distinct license plates is $5 \cdot 10^4 \cdot 26^2$.

2013B

18. **Answer (D):** First note that the only number between 2000 and 2013 that shares this property is 2002.

Consider now the numbers in the range 1001 to 1999. There is exactly 1 number, 1001, that shares the property when the units digits is 1. There are exactly 2 numbers, 1102 and 1012, when the units digit is 2; exactly 3 numbers, 1203, 1113, and 1023, when the units digits is 3, and so on. Because the thousands digit is always 1, when the units digit is n, for $1 \leq n \leq 9$, the sum of the hundreds and tens digits must be $n - 1$. There are exactly n ways for this to occur. Hence there are exactly

$$1 + (1 + 2 + \cdots + 9) = 1 + \frac{9 \cdot 10}{2} = 1 + 45 = 46$$

numbers that share this property.

2016B

18. **Answer (E):** A sum of consecutive integers is equal to the number of integers in the sum multiplied by their median. Note that $345 = 3 \cdot 5 \cdot 23$. If there are an odd number of integers in the sum, then the median and the number of integers must be complementary factors of 345. The only possibilities are 3 integers with median $5 \cdot 23 = 115$, 5 integers with median $3 \cdot 23 = 69$, $3 \cdot 5 = 15$ integers with median 23, and 23 integers with median $3 \cdot 5 = 15$. Having more integers in the sum would force some of the integers to be negative. If there are an even number of integers in the sum, say $2k$, then the median will be $\frac{j}{2}$, where k and j are complementary factors of 345. The possibilities are 2 integers with median $\frac{345}{2}$, 6 integers with median $\frac{115}{2}$, and 10 integers with median $\frac{69}{2}$. Again, having more integers in the sum would force some of the integers to be negative. This gives a total of 7 solutions.

2018A 18. **Answer (D):** Let S be the set of integers, both negative and nonnegative, having the given form. Increasing the value of a_i by 1 for $0 \le i \le 7$ creates a one-to-one correspondence between S and the ternary (base 3) representation of the integers from 0 through $3^8 - 1$, so S contains $3^8 = 6561$ elements. One of those is 0, and by symmetry, half of the others are positive, so S contains $1 + \frac{1}{2} \cdot (6561 - 1) = 3281$ elements.

OR

First note that if an integer N can be written in this form, then $N - 1$ can also be written in this form as long as not all the a_i in the representation of N are equal to -1. A procedure to alter the representation of N so that it will represent $N - 1$ instead is to find the least value of i such that $a_i \ne -1$, reduce the value of that a_i by 1, and set $a_i = 1$ for all lower values of i. By the formula for the sum of a finite geometric series, the greatest integer that can be written in the given form is

$$\frac{3^8 - 1}{3 - 1} = 3280.$$

Therefore, 3281 nonnegative integers can be written in this form, namely all the integers from 0 through 3280, inclusive. (The negative integers from -3280 through -1 can also be written in this way.)

OR

Think of the indicated sum as an expansion in base 3 using "digits" -1, 0, and 1. Note that the leftmost digit a_k of any positive integer that can be written in this form cannot be negative and therefore must be 1. Then there are 3 choices for each of the remaining k digits to the right of a_k, resulting in 3^k positive integers that can be written in the indicated form. Thus there are

$$\sum_{k=0}^{7} 3^k = \frac{3^8 - 1}{3 - 1} = 3280$$

positive numbers of the indicated form. Because 0 can also be written in this form, the number of nonnegative integers that can be written in the indicated form is 3281.

2018B

18. **Answer (D):** Let X, Y, and Z denote the three different families in some order. Then the only possible arrangements are to have the second row be members of XYZ and the third row be members of ZXY, or to have the second row be members of XYZ and the third row be members of YZX. Note that these are not the same, because in the first case one sibling pair occupy the right-most seat in the second row and the left-most seat in the third row, whereas in the second case this does not happen. (Having members of XYX in the second row does not work because then the third row must be members of ZYZ to avoid consecutive members of Z; but in this case one of the Y siblings would be seated directly in front of the other Y sibling.) In each of these 2 cases there are $3! = 6$ ways to assign the families to the letters and $2^3 = 8$ ways to position the boy and girl within the seats assigned to the families. Therefore the total number of seating arrangements is $2 \cdot 6 \cdot 8 = 96$.

2001

19. **(D)** The number of possible selections is the number of solutions of the equation

$$g + c + p = 4$$

where g,c, and p represent, respectively, the number of glazed, chocolate, and powdered donuts. The 15 possible solutions to this equations are (4,0,0), (0,4,0), (0,0,4), (3,0,1),(3,1,0),(1,3,0),(0,3,1),(1,0,3),(0,1,3),(2,2,0),(2,0,2), (0,2,2), (2,1,1),(1 and (1,1,2).

OR

Code each selection as a sequence of four *'s and two —'s, where * represents a donut and each — denotes a "separator" between types of donuts. For example **—*—* represents two glazed donuts, one chocolate donut, and one powdered donut. From the six slots that can be occupied by a — or a *, we must choose two places for the —'s to determine a selection. Thus, there are $\binom{6}{2} \equiv C_2^6 \equiv 6C2 = 15$ selections.

2009A

19. **Answer (B):** Circles A and B have circumferences 200π and $2\pi r$, respectively. After circle B begins to roll, its initial point of tangency with circle A touches circle A again a total of

$$\frac{200\pi}{2\pi r} = \frac{100}{r}$$

times. In order for this to be an integer greater than 1, r must be one of the integers 1, 2, 4, 5, 10, 20, 25, or 50. Hence there are a total of 8 possible values of r.

2017A

19. **Answer (C):** Let X be the set of ways to seat the five people in which Alice sits next to Bob. Let Y be the set of ways to seat the

five people in which Alice sits next to Carla. Let Z be the set of ways to seat the five people in which Derek sits next to Eric. The required answer is $5! - |X \cup Y \cup Z|$. The Inclusion–Exclusion Principle gives

$$|X \cup Y \cup Z| = (|X|+|Y|+|Z|)-(|X \cap Y|+|X \cap Z|+|Y \cap Z|)+|X \cap Y \cap Z|.$$

Viewing Alice and Bob as a unit in which either can sit on the other's left side shows that there are $2 \cdot 4! = 48$ elements of X. Similarly there are 48 elements of Y and 48 elements of Z. Viewing Alice, Bob, and Carla as a unit with Alice in the middle shows that $|X \cap Y| = 2 \cdot 3! = 12$. Viewing Alice and Bob as a unit and Derek and Eric as a unit shows that $|X \cap Z| = 2 \cdot 2 \cdot 3! = 24$. Similarly $|Y \cap Z| = 24$. Finally, there are $2 \cdot 2 \cdot 2! = 8$ elements of $X \cap Y \cap Z$. Therefore $|X \cup Y \cup Z| = (48+48+48)-(12+24+24)+8 = 92$, and the answer is $120 - 92 = 28$.

OR

There are three cases based on where Alice is seated.

- If Alice takes the first or last chair, then Derek or Eric must be seated next to her, Bob or Carla must then take the middle chair, and either of the remaining two individuals can be seated in either of the other two chairs. This gives a total of $2^4 = 16$ arrangements.

- If Alice is seated in the second or fourth chair, then Derek and Eric will take the seats on her two sides, and this can be done in two ways. Bob and Carla can be seated in the two remaining chairs in two ways, which yields a total of $2^3 = 8$ arrangements.

- If Alice sits in the middle chair, then Derek and Eric will be seated on her two sides, with Bob and Carla seated in the first and last chairs. This results in $2^2 = 4$ arrangements.

Thus there are $16 + 8 + 4 = 28$ possible arrangements in total.

2007B

20. **Answer (C):** After one of the 25 blocks is chosen, 16 of the remaining blocks do not share its row or column. After the second block is chosen, 9 of the remaining blocks do not share a row or column with either of the first two. Because the three blocks can be chosen in any order, the number of different combinations is

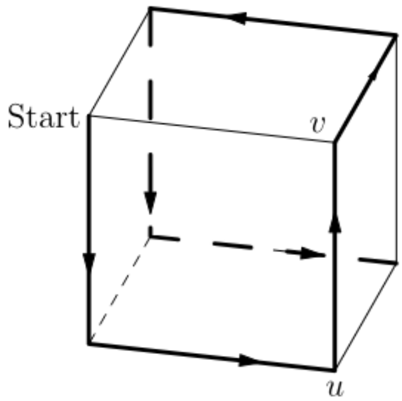
$$\frac{25 \cdot 16 \cdot 9}{3!} = 25 \cdot 8 \cdot 3 = 600.$$

2015A

20. **Answer (B):** Let x and y be the lengths of the sides of the rectangle. Then $A+P = xy+2x+2y = (x+2)(y+2)-4$, so $A+P+4$ must be the product of two factors, each of which is greater than 2. Because the only factorization of $102+4=106$ into two factors greater than 1 is $2 \cdot 53$, $A+P$ cannot equal 102. Because $100+4=104=4 \cdot 26$, $104+4=108=3 \cdot 36$, $106+4=110=5 \cdot 22$, and $108+4=112=4 \cdot 28$, the other choices equal $A+P$ for rectangles with dimensions 2×24, 1×34, 3×20, and 2×26, respectively.

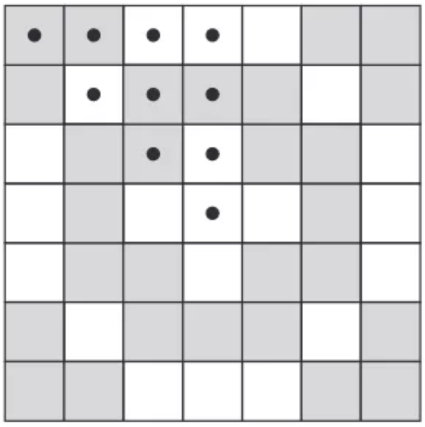
2015B

20. **Answer (A):** The first two edges of Erin's crawl can be chosen in $3 \cdot 2 = 6$ ways. These edges share a unique face of the cube, called the initial face. At this point, Erin is standing at a vertex u and there is only one unvisited vertex v of the initial face. If v is not visited right after u, then Erin visits all vertices adjacent to v before v. This means that once Erin reaches v, she cannot continue her crawl to any unvisited vertex, and v cannot be her last visited vertex because v is adjacent to her starting point. Thus v must be visited right after u. There are only two ways to visit the remaining four vertices (clockwise or counterclockwise around the face opposite to the initial face) and exactly one of them cannot be followed by a return to the starting vertex. Therefore there are exactly 6 paths in all.



2018A

20. **Answer (B):** None of the squares that are marked with dots in the sample scanning code shown below can be mapped to any other marked square by reflections or non-identity rotations. Therefore these 10 squares can be arbitrarily colored black or white in a symmetric scanning code, with the exception of "all black" and "all white". On the other hand, reflections or rotations will map these squares to all the other squares in the scanning code, so once these 10 colors are specified, the symmetric scanning code is completely determined. Thus there are $2^{10} - 2 = 1022$ symmetric scanning codes.



OR

The diagram below shows the orbits of each square under rotations and reflections. Because the scanning code must look the same under these transformations, all squares in the same orbit must get the same color, but one is free to choose the color for each orbit, except for the choice of "all black" and "all white". Because there are 10 orbits, there are $2^{10} - 2 = 1022$ symmetric scanning codes.

A	*B*	*C*	*D*	*C*	*B*	*A*
B	*E*	*F*	*G*	*F*	*E*	*B*
C	*F*	*H*	*I*	*H*	*F*	*C*
D	*G*	*I*	*J*	*I*	*G*	*D*
C	*F*	*H*	*I*	*H*	*F*	*C*
B	*E*	*F*	*G*	*F*	*E*	*B*
A	*B*	*C*	*D*	*C*	*B*	*A*