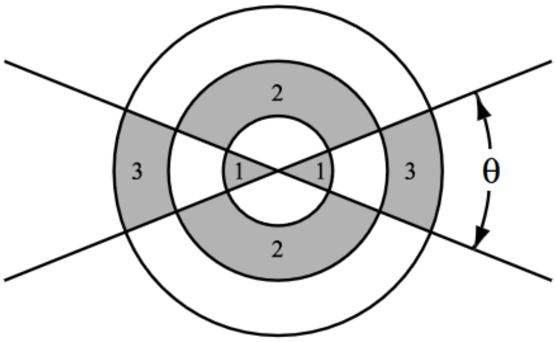
1

2D GEOMETRY

2004A 21. **(B)** Let θ be the acute angle between the two lines. The area of shaded Region 1 in the diagram is

$$2\left(\frac{1}{2}\theta(1)^2\right) = \theta.$$



The area of shaded Region 2 is

$$2\left(\frac{1}{2}(\pi - \theta)(2^2 - 1^2)\right) = 3\pi - 3\theta.$$

The area of shaded Region 3 is

$$2\left(\frac{1}{2}\theta(3^2 - 2^2)\right) = 5\theta.$$

Hence the total area of the shaded regions is $3\pi + 3\theta$. The area bounded by the largest circle is 9π, so

$$\frac{3\pi + 3\theta}{9\pi} = \frac{8}{8 + 13}.$$

Solving for θ gives $\theta = \pi/7$.

2007B

21. **Answer (B):** Let s be the side length of the square, and let h be the length of the altitude of $\triangle ABC$ from B. Because $\triangle ABC$ and $\triangle WBZ$ are similar, it follows that

$$\frac{h-s}{s} = \frac{h}{AC} = \frac{h}{5}, \quad \text{so} \quad s = \frac{5h}{5+h}.$$

Because $h = 3 \cdot 4/5 = 12/5$, the side length of the square is

$$s = \frac{5(12/5)}{5+12/5} = \frac{60}{37}.$$

OR

Because $\triangle WBZ$ is similar to $\triangle ABC$, we have

$$BZ = \frac{4}{5}s \quad \text{and} \quad CZ = 4 - \frac{4}{5}s.$$

Because $\triangle ZYC$ is similar to $\triangle ABC$, we have

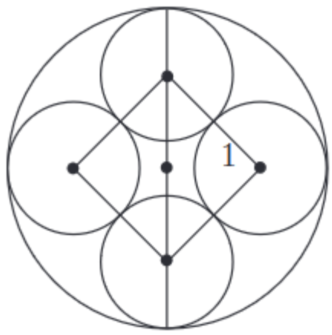
$$\frac{s}{4-(4/5)s} = \frac{3}{5}.$$

Thus

$$5s = 12 - \frac{12}{5}s \quad \text{and} \quad s = \frac{60}{37}.$$

2009A

21. **Answer (C):** It may be assumed that the smaller circles each have radius 1. Their centers form a square with side length 2 and diagonal length $2\sqrt{2}$. Thus the diameter of the large circle is $2+2\sqrt{2}$, so its area is $(1+\sqrt{2})^2\pi = (3+2\sqrt{2})\pi$. The desired ratio is



$$\frac{4\pi}{(3+2\sqrt{2})\pi} = 4(3-2\sqrt{2}).$$

2003A

22. (B) We have $EA = 5$ and $CH = 3$. Triangles GCH and GEA are similar, so

$$\frac{GC}{GE} = \frac{3}{5} \quad \text{and} \quad \frac{CE}{GE} = \frac{GE - GC}{GE} = 1 - \frac{3}{5} = \frac{2}{5}.$$

Triangles GFE and CDE are similar, so

$$\frac{GF}{8} = \frac{CE}{GE} = \frac{5}{2}$$

and $FG = 20$.

OR

Place the figure in the coordinate plane with the origin at D, $\overline{DA}$ on the positive x-axis, and $\overline{DC}$ on the positive y-axis. Then $H = (3, 8)$ and $A = (9, 0)$, so line AG has the equation

$$y = -\frac{4}{3}x + 12.$$

Also, $C = (0, 8)$ and $E = (4, 0)$, so line EG has the equation

$$y = -2x + 8.$$

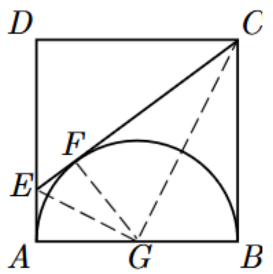
The lines intersect at $(-6, 20)$, so $FG = 20$.

2004A

22. **(D)** Let F be the point at which $\overline{CE}$ is tangent to the semicircle, and let G be the midpoint of $\overline{AB}$. Because $\overline{CF}$ and $\overline{CB}$ are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = AE$. The Pythagorean Theorem applied to $\triangle CDE$ gives

$$(2 - x)^2 + 2^2 = (2 + x)^2.$$

It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



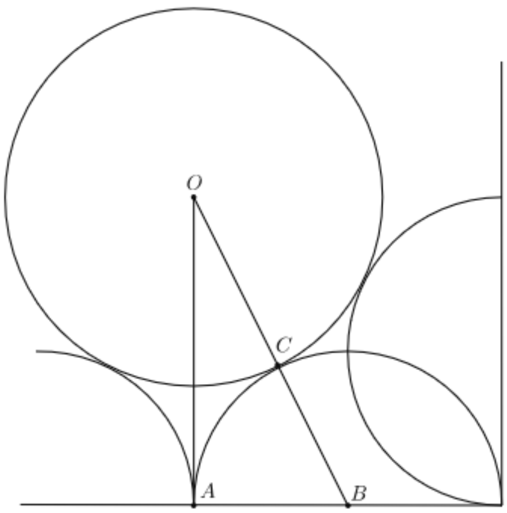
2009B

22. **Answer (B):** The area of triangle A is 1, and its hypotenuse has length $\sqrt{5}$. Triangle B is similar to triangle A and has a hypotenuse of 2, so its area is $(\frac{2}{\sqrt{5}})^2 = \frac{4}{5}$. The volume of the required piece is $c = \frac{4}{5} \cdot 2 = \frac{8}{5}$ cubic inches. The icing on this piece has an area of $s = \frac{4}{5} + 2^2 = \frac{24}{5}$ square inches. Therefore $c + s = \frac{8}{5} + \frac{24}{5} = \frac{32}{5}$.

2013B

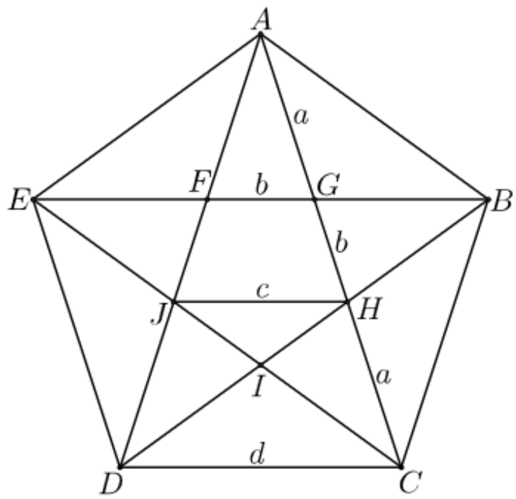
22. **Answer (C):** The digit j at J contributes to all four sums, and each of the other digits contributes to exactly one sum. Therefore the sum of all four sums is $3j + (1 + 2 + 3 + \cdots + 9) = 45 + 3j$. Because all four sums are equal, this must be a multiple of 4, so $j = 1$, 5, or 9. For each choice of j, pair up the remaining digits so that each pair has the same sum. For example, for $j = 1$ the pairs are 2 and 9, 3 and 8, 4 and 7, and 5 and 6. Then order the pairs so that they correspond to the vertex pairs (A, E), (B, F), (C, G), (D, H). This results in $2^4 \cdot 4!$ different combinations for each j. Thus the requirements can be met in $2^4 \cdot 4! \cdot 3 = 1152$ ways.

2014B 22. **Answer (B):** Let O be the center of the circle and choose one of the semicircles to have center point B. Label the point of tangency C and point A as in the figure. In $\triangle OAB$, $AB = \frac{1}{2}$ and $OA = 1$, so $OB = \frac{\sqrt{5}}{2}$. Because $BC = \frac{1}{2}$, $OC = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}$.



2015B 22. **Answer (D):** Triangles AGB and CHJ are isosceles and congruent, so $AG = HC = HJ = 1$. Triangles AFG and BGH are congruent, so $FG = GH$. Triangles AGF, AHJ, and ACD are similar, so $\frac{a}{b} = \frac{a+b}{c} = \frac{2a+b}{d}$.

Because $a = c = 1$, the first equation becomes $\frac{1}{b} = \frac{1+b}{1}$ or $b^2 + b - 1 = 0$, so $b = \frac{-1+\sqrt{5}}{2}$. Substituting this in the second equation gives $d = \frac{1+\sqrt{5}}{2}$, so $b + c + d = 1 + \sqrt{5}$.



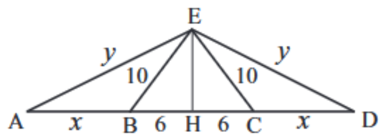
2002A

23. **(D)** Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles. Segment $\overline{EH}$ is the common altitude of the two isosceles triangles $\triangle AED$ and $\triangle BEC$, and

$$EH = \sqrt{10^2 - 6^2} = 8.$$

Let $AB = CD = x$ and $AE = ED = y$. Then $2x+2y+12 = 2(32)$, so $y = 26-x$. Thus,

$$8^2 + (x+6)^2 = y^2 = (26-x)^2 \quad \text{and} \quad x = 9.$$



2003A

23. (C) The base row of the large equilateral triangle has 1001 triangles pointing downward and 1002 pointing upward. This base row requires 3(1002) toothpicks since the downward pointing triangles require no additional toothpicks. Each succeeding row will require one less set of 3 toothpicks, so the total number of toothpicks required is

$$3(1002 + 1001 + 1000 + \cdots + 2 + 1) = 3 \cdot \frac{1002 \cdot 1003}{2} = 1,507,509.$$

OR

Create a table:

Number of Rows	Number of Triangles in Base Row	Number of Toothpicks in All Rows
1	1	3
2	3	$3+6$
3	5	$3+6+9$
$\vdots$	$\vdots$	$\vdots$
n	$2n-1$	$3(1+2+\cdots+n)$

Thus

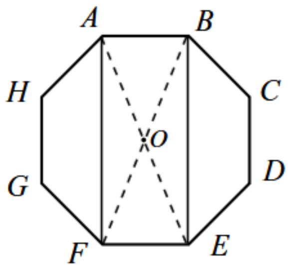
$$2003 = 2n - 1 \quad \text{so} \quad n = 1002.$$

The number of toothpicks is

$$3(1 + 2 + \cdots + 1002) = 3\frac{(1002)(1003)}{2} = 1,507,509.$$

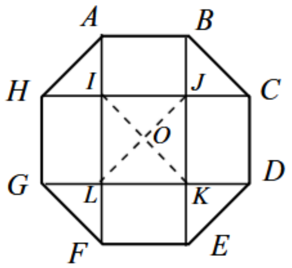
2003B

23. (D) Let O be the intersection of the diagonals of $ABEF$. Since the octagon is regular, $\triangle AOB$ has area $1/8$. Since O is the midpoint of $\overline{AE}$, $\triangle OAB$ and $\triangle BOE$ have the same area. Thus $\triangle ABE$ has area $1/4$, so $ABEF$ has area $1/2$.



OR

Let O be the intersection of the diagonals of the square $IJKL$. Rectangles $ABJI$, $JCDK$, $KEFL$, and $LGHI$ are congruent. Also $IJ = AB = AH$, so the right isosceles triangles $\triangle AIH$ and $\triangle JOI$ are congruent. By symmetry, the area in the center square $IJKL$ is the sum of the areas of $\triangle AIH$, $\triangle CJB$, $\triangle EKD$, and $\triangle GLF$. Thus the area of rectangle $ABEF$ is half the area of the octagon.



2004A

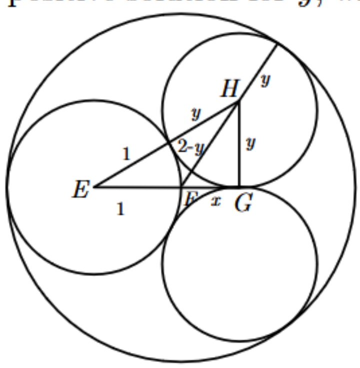
23. **(D)** Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$. Since the center of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2, which is the diameter of circle A. Applying the Pythagorean Theorem to right triangles EGH and FGH gives

$$(1+y)^2 = (1+x)^2 + y^2 \quad \text{and} \quad (2-y)^2 = x^2 + y^2,$$

from which it follows that

$$y = x + \frac{x^2}{2} \quad \text{and} \quad y = 1 - \frac{x^2}{4}.$$

The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (-2, 0)$. The radius of circle B is the positive solution for y, which is $8/9$.

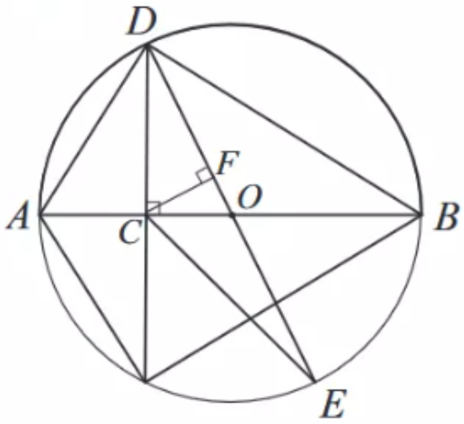


2005A

23. **(C)** Let O be the center of the circle. Each of $\triangle DCE$ and $\triangle ABD$ has a diameter of the circle as a side. Thus the ratio of their areas is the ratio of the two altitudes to the diameters. These altitudes are $\overline{DC}$ and the altitude from C to $\overline{DO}$ in $\triangle DCE$. Let F be the foot of this second altitude. Since $\triangle CFO$ is similar to $\triangle DCO$,

$$\frac{CF}{DC} = \frac{CO}{DO} = \frac{AO - AC}{DO} = \frac{\frac{1}{2}AB - \frac{1}{3}AB}{\frac{1}{2}AB} = \frac{1}{3},$$

which is the desired ratio.



OR

Because $AC = AB/3$ and $AO = AB/2$, we have $CO = AB/6$. Triangles DCO and DAB have a common altitude to $\overline{AB}$ so the area of $\triangle DCO$ is $\frac{1}{6}$ the area of $\triangle ADB$. Triangles DCO and ECO have equal areas since they have a common base $\overline{CO}$ and their altitudes are equal. Thus the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$ is $1/3$.

2006A

23. **(B)** Radii $\overline{AC}$ and $\overline{BD}$ are each perpendicular to $\overline{CD}$. By the Pythagorean Theorem,

$$CE = \sqrt{5^2 - 3^2} = 4.$$

Because $\triangle ACE$ and $\triangle BDE$ are similar,

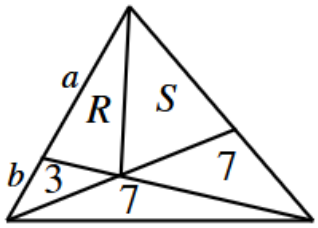
$$\frac{DE}{CE} = \frac{BD}{AC}, \quad \text{so} \quad DE = CE \cdot \frac{BD}{AC} = 4 \cdot \frac{8}{3} = \frac{32}{3}.$$

Therefore

$$CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$$

2006B

23. **(D)** Partition the quadrilateral into two triangles and let the areas of the triangles be R and S as shown. Then the required area is $T = R + S$.



Let a and b, respectively, be the bases of the triangles with areas R and 3, as indicated. If two triangles have the same altitude, then the ratio of their areas is the same as the ratio of their bases. Thus

$$\frac{a}{b} = \frac{R}{3} = \frac{R+S+7}{3+7}, \quad \text{so} \quad \frac{R}{3} = \frac{T+7}{10}.$$

Similarly,

$$\frac{S}{7} = \frac{S+R+3}{7+7}, \quad \text{so} \quad \frac{S}{7} = \frac{T+3}{14}.$$

Thus

$$T = R + S = 3\left(\frac{T+7}{10}\right) + 7\left(\frac{T+3}{14}\right).$$

From this we obtain

$$10T = 3(T+7) + 5(T+3) = 8T + 36,$$

and it follows that $T = 18$.

2014A

23. **Answer (C):** Without loss of generality, assume that the rectangle has dimensions 3 by $\sqrt{3}$. Then the fold has length 2, and the overlapping areas are equilateral triangles each with area $\frac{\sqrt{3}}{4} \cdot 2^2$. The new shape has area $3\sqrt{3} - \frac{\sqrt{3}}{4} \cdot 2^2 = 2\sqrt{3}$, and the desired ratio is $2\sqrt{3} : 3\sqrt{3} = 2 : 3$.

2018A

23. **Answer (D):** Let the triangle's vertices in the coordinate plane be $(4, 0)$, $(0, 3)$, and $(0, 0)$, with $[0, s] \times [0, s]$ representing the unplanted portion of the field. The equation of the hypotenuse is $3x+4y-12 = 0$, so the distance from (s, s), the corner of S closest to the hypotenuse, to this line is given by

$$\frac{|3s + 4s - 12|}{\sqrt{3^2 + 4^2}}.$$

Setting this equal to 2 and solving for s gives $s = \frac{22}{7}$ and $s = \frac{2}{7}$, and the former is rejected because the square must lie within the triangle. The unplanted area is thus $\left(\frac{2}{7}\right)^2 = \frac{4}{49}$, and the requested fraction is

$$1 - \frac{\frac{4}{49}}{\frac{1}{2} \cdot 4 \cdot 3} = \frac{145}{147}.$$

OR

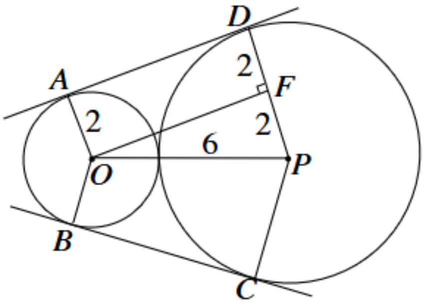
Let the given triangle be described as $\triangle ABC$ with the right angle at B and $AB = 3$. Let D be the vertex of the square that is in the interior of the triangle, and let s be the edge length of the square. Then two sides of the square along with line segments $\overline{AD}$ and $\overline{CD}$ decompose $\triangle ABC$ into four regions. These regions are a triangle with base 5 and height 2, the unplanted square with side s, a right triangle with legs s and $3 - s$, and a right triangle with legs s and $4 - s$. The sum of the areas of these four regions is

$$\frac{1}{2} \cdot 5 \cdot 2 + s^2 + \frac{1}{2}s(3 - s) + \frac{1}{2}s(4 - s) = 5 + \frac{7}{2}s,$$

and the area of $\triangle ABC$ is 6. Solving $5 + \frac{7}{2}s = 6$ for s gives $s = \frac{2}{7}$, and the solution concludes as above.

2006B

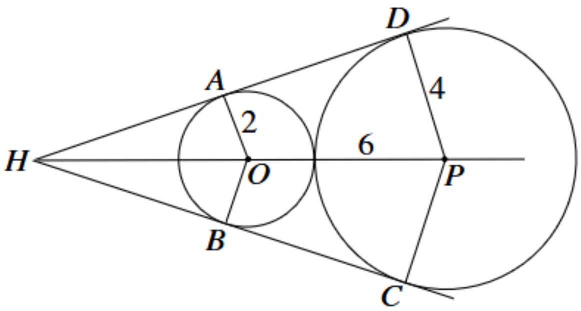
24. **(B)** Through O draw a line parallel to $\overline{AD}$ intersecting $\overline{PD}$ at F.



Then $AOFD$ is a rectangle and OPF is a right triangle. Thus $DF = 2$, $FP = 2$, and $OF = 4\sqrt{2}$. The area of trapezoid $AOPD$ is $12\sqrt{2}$, and the area of hexagon $AOBCPD$ is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

OR

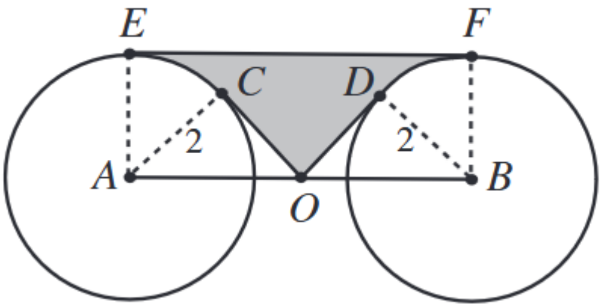
Lines $AD, BC,$ and OP intersect at a common point H.



Because $\angle PDH = \angle OAH = 90°$, triangles PDH and OAH are similar with ratio of similarity 2. Thus $2HO = HP = HO + OP = HO + 6$, so $HO = 6$ and $AH = \sqrt{HO^2 - OA^2} = 4\sqrt{2}$. Hence the area of $\triangle OAH$ is $(1/2)(2)(4\sqrt{2}) = 4\sqrt{2}$, and the area of $\triangle PDH$ is $(2^2)(4\sqrt{2}) = 16\sqrt{2}$. The area of the hexagon is twice the area of $\triangle PDH$ minus twice the area of $\triangle OAH$, so it is $24\sqrt{2}$.

2007A

24. **Answer (B):** Rectangle $ABFE$ has area $AE \cdot AB = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$. Right triangles ACO and BDO each have hypotenuse $2\sqrt{2}$ and one leg of length 2.



Hence they are each isosceles, and each has area $(1/2)\left(2^2\right) = 2$. Angles CAE and DBF are each $45°$, so sectors CAE and DBF each have area

$$\frac{1}{8} \cdot \pi \cdot 2^2 = \frac{\pi}{2}.$$

Thus the area of the shaded region is

$$8\sqrt{2} - 2 \cdot 2 - 2 \cdot \frac{\pi}{2} = 8\sqrt{2} - 4 - \pi.$$

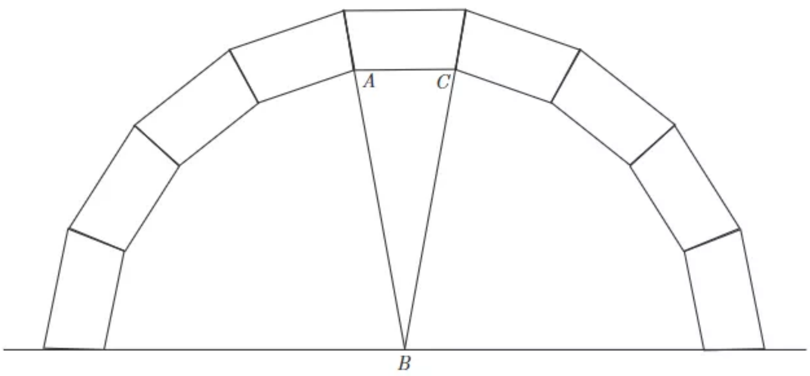
2009B

24. **Answer (A):** Add a symmetric arch to the given arch to create a closed loop of trapezoids. Consider the regular 18-sided polygon created by the interior of the completed loop. Each interior angle of a regular 18-gon measures

$$(18-2)\cdot 180°/18 = 160°.$$

Then $x + x + 160° = 360°$, so $x = 100°$.

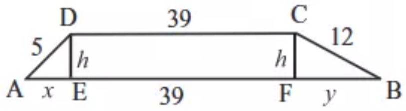
OR



Extend two sides of a trapezoid until they meet at the center of the arch, as shown. Then $\triangle ABC$ is isosceles and by symmetry $\angle ABC = \frac{180}{9} = 20°$, and $\angle BAC = 80°$. The requested angle is supplementary to $\angle BAC$, so $x = 180 - 80 = 100°$.

2002A 25. **(C)** First drop perpendiculars from D and C to $\overline{AB}$. Let E and F be the feet of the perpendiculars to $\overline{AB}$ from D and C, respectively, and let

$$h = DE = CF, \quad x = AE, \quad \text{and} \quad y = FB.$$



Then

$$25 = h^2 + x^2, \quad 144 = h^2 + y^2, \quad \text{and} \quad 13 = x + y.$$

So

$$144 = h^2 + y^2 = h^2 + (13 - x)^2 = h^2 + x^2 + 169 - 26x = 25 + 169 - 26x,$$

which gives $x = 50/26 = 25/13$, and

$$h = \sqrt{5^2 - \left(\frac{25}{13}\right)^2} = 5\sqrt{1 - \frac{25}{169}} = 5\sqrt{\frac{144}{169}} = \frac{60}{13}.$$

Hence

$$\text{Area } (ABCD) = \frac{1}{2}(39 + 52) \cdot \frac{60}{13} = 210.$$

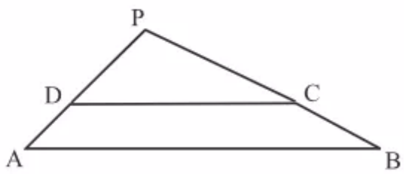
OR

Extend $\overline{AD}$ and $\overline{BC}$ to intersect at P. Since $\triangle PDC$ and $\triangle PAB$ are similar, we have

$$\frac{PD}{PD + 5} = \frac{39}{52} = \frac{PC}{PC + 12}.$$

So $PD = 15$ and $PC = 36$. Note that 15, 36, and 39 are three times 5, 12, and 13, respectively, so $\angle APB$ is a right angle. The area of the trapezoid is the difference of the areas of $\triangle PAB$ and $\triangle PDC$, so

$$\text{Area}(ABCD) = \frac{1}{2}(20)(48) - \frac{1}{2}(15)(36) = 210.$$



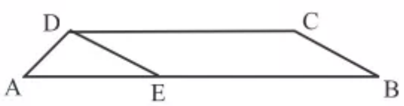
OR

Draw the line through D parallel to $\overline{BC}$, intersecting $\overline{AB}$ at E. Then $BCDE$ is a parallelogram, so $DE = 12$, $EB = 39$, and $AE = 52 - 39 = 13$. Thus $DE^2 + AD^2 = AE^2$, and $\triangle ADE$ is a right triangle. Let h be the altitude from D to $\overline{AE}$, and note that

$$\text{Area}(ADE) = \frac{1}{2}(5)(12) = \frac{1}{2}(13)(h),$$

so $h = 60/13$. Thus

$$\text{Area}(ABCD) = \frac{60}{13} \cdot \frac{1}{2}(39 + 52) = 210.$$



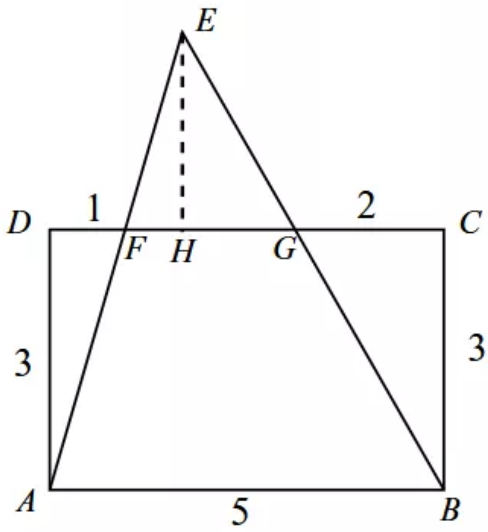
2004B

20. (D) Let H be the foot of the perpendicular from E to $\overline{DC}$. Since $CD = AB = 5$, $FG = 2$, and $\triangle FEG$ is similar to $\triangle AEB$, we have

$$\frac{EH}{EH+3} = \frac{2}{5}, \quad \text{so} \quad 5EH = 2EH + 6,$$

and $EH = 2$. Hence

$$\text{Area}(\triangle AEB) = \frac{1}{2}(2+3)\cdot 5 = \frac{25}{2}.$$



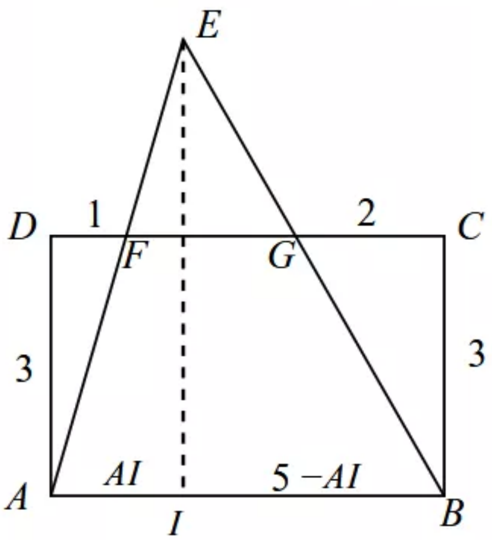
OR

Let I be the foot of the perpendicular from E to $\overline{AB}$. Since

$$\triangle EIA \text{ is similar to } \triangle ADF \quad \text{and} \quad \triangle EIB \text{ is similar to } \triangle BCG,$$

we have

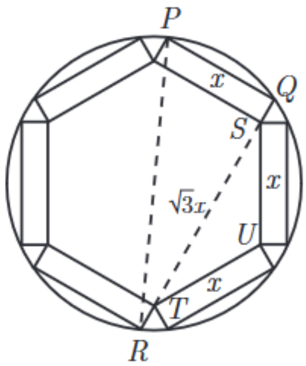
$$\frac{AI}{EI} = \frac{1}{3} \quad \text{and} \quad \frac{5-AI}{EI} = \frac{2}{3}.$$



Adding gives $5/EI = 1$, so $EI = 5$. The area of the triangle is $\frac{1}{2}\cdot 5\cdot 5 = \frac{25}{2}$.

2008A

25. **Answer (C):** Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and $\triangle PQR$ is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then $\triangle STU$ is an isosceles triangle with two sides of length x and vertex angle 120°. It follows that $ST = \sqrt{3}x$, so $QR = QS+ST+TR = \sqrt{3}x+2$. Apply the Pythagorean Theorem to $\triangle PQR$ to obtain $\left(\sqrt{3}x+2\right)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain



$$x = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$