10

PROBABILITY

2003B

21. (C) The beads will all be red at the end of the third draw precisely when two green beads are chosen in the three draws. If the first bead drawn is green, then there will be one green and three red beads in the bag before the second draw. So the probability that green beads are drawn in the first two draws is

$$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

The probability that a green bead is chosen, then a red bead, and then a green bead, is

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{32}.$$

Finally, the probability that a red bead is chosen then two green beads is

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{16}.$$

The sum of these probabilities is

$$\frac{1}{8} + \frac{3}{32} + \frac{1}{16} = \frac{9}{32}.$$

2005B

21. **(A)** The total number of ways that the numbers can be chosen is $\binom{40}{4}$. Exactly 10 of these possibilities result in the four slips having the same number.

Now we need to determine the number of ways that two slips can have a number a and the other two slips have a number b, with $b \neq a$. There are $\binom{10}{2}$ ways to

choose the distinct numbers a and b. For each value of a there are $\binom{4}{2}$ to choose the two slips with a and for each value of b there are $\binom{4}{2}$ to choose the two slips with b. Hence the number of ways that two slips have some number a and the other two slips have some distinct number b is

$$\binom{10}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} = 45 \cdot 6 \cdot 6 = 1620.$$

So the probabilities q and p are $\frac{10}{\binom{40}{4}}$ and $\frac{1620}{\binom{40}{4}}$, respectively, which implies that

$$\frac{p}{q} = \frac{1620}{10} = 162.$$

2006B

21. **(C)** On each die the probability of rolling k, for $1 \leq k \leq 6$, is

$$\frac{k}{1+2+3+4+5+6} = \frac{k}{21}.$$

There are six ways of rolling a total of 7 on the two dice, represented by the ordered pairs $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Thus the probability of rolling a total of 7 is

$$\frac{1 \cdot 6 + 2 \cdot 5 + 3 \cdot 4 + 4 \cdot 3 + 5 \cdot 2 + 6 \cdot 1}{21^2} = \frac{56}{21^2} = \frac{8}{63}.$$

2010B

21. **Answer (E):** Each four-digit palindrome has digit representation $abba$ with $1 \le a \le 9$ and $0 \le b \le 9$. The value of the palindrome is $1001a + 110b$. Because 1001 is divisible by 7 and 110 is not, the palindrome is divisible by 7 if and only if $b = 0$ or $b = 7$. Thus the requested probability is $\frac{2}{10} = \frac{1}{5}$.

2011A

21. **Answer (D):** The weights of the two pairs of coins are equal if each pair contains the same number of counterfeit coins. Therefore either the first pair and the second pair both contain only genuine coins, or the first pair and the second pair both contain one counterfeit coin. The number of ways to choose the coins in the first case is $\binom{8}{2} \cdot \binom{6}{2} = 420$. The number of ways to choose the coins in the second case is $8 \cdot 2 \cdot 7 \cdot 1 = 112$. Therefore the requested probability is $\frac{420}{112+420} = \frac{15}{19}$.

2007B

22. **Answer (B):** The probability of the number appearing 0, 1, and 2 times is

$$P(0) = \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}, \quad P(1) = 2 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{6}{16}, \quad \text{and} \quad P(2) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16},$$

respectively. So the expected return, in dollars, to the player is

$$P(0) \cdot (-1) + P(1) \cdot (1) + P(2) \cdot (2) = \frac{-9 + 6 + 2}{16} = -\frac{1}{16}.$$

2008B

22. **Answer (C):** There are $6!/(3!2!1!) = 60$ distinguishable orders of the beads on the line. To meet the required condition, the red beads must be placed in one of four configurations: positions 1, 3, and 5, positions 2, 4, and 6, positions 1, 3, and 6, or positions 1, 4, and 6. In the first two cases, the blue bead can be placed in any of the three remaining positions. In the last two cases, the blue bead can be placed in either of the two adjacent remaining positions. In each case, the placement of the white beads is then determined. Hence there are $2 \cdot 3 + 2 \cdot 2 = 10$ orders that meet the required condition, and the requested probability is $\frac{10}{60} = \frac{1}{6}$.

2009A

22. **Answer (D):** Suppose that the two dice originally had the numbers 1, 2, 3, 4, 5, 6 and $1'$, $2'$, $3'$, $4'$, $5'$, $6'$, respectively. The process of randomly picking the numbers, randomly affixing them to the dice, rolling the dice, and adding the top numbers is equivalent to picking two of the twelve numbers at random and adding them. There are $\binom{12}{2} = 66$ sets of two elements taken from $S = \{1, 1', 2, 2', 3, 3', 4, 4', 5, 5', 6, 6'\}$. There are 4 ways to use a 1 and 6 to obtain 7, namely, $\{1, 6\}$, $\{1, 6'\}$, $\{1', 6\}$, and $\{1', 6'\}$. Similarly there are 4 ways to obtain the sum of 7 using a 2 and 5, and 4 ways using a 3 and 4. Hence there are 12 pairs taken from S whose sum is 7. Therefore the requested probability is $\frac{12}{66} = \frac{2}{11}$.

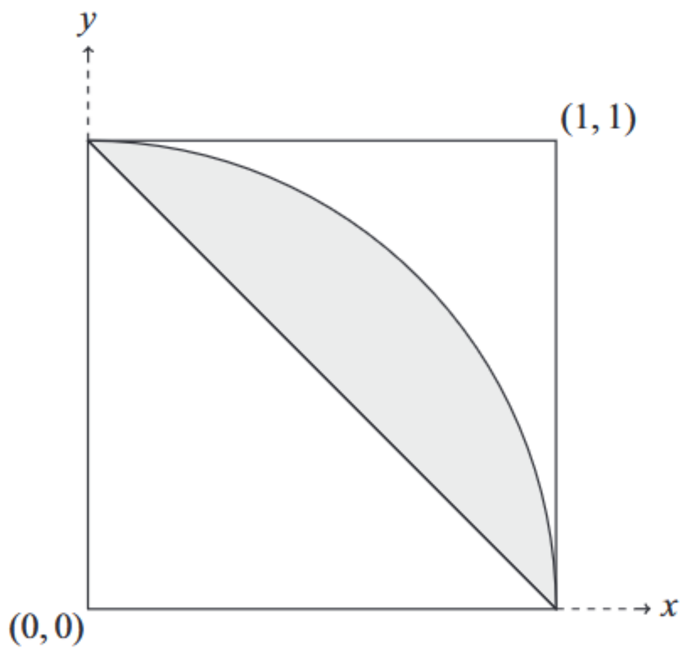
OR

Because the process is equivalent to picking two of the twelve numbers at random and then adding them, suppose we first pick number N. Then the second choice must be number $7 - N$. For any value of N, there are two "removable numbers" equal to $7 - N$ out of the remaining 11, so the probability of rolling a 7 is $\frac{2}{11}$.

2015A

22. **Answer (A):** There are $2^8 = 256$ equally likely outcomes of the coin tosses. Classify the possible arrangements around the table according to the number of heads flipped. There is 1 possibility with no heads, and there are 8 possibilities with exactly one head. There are $\binom{8}{2} = 28$ possibilities with exactly two heads, 8 of which have two adjacent heads. There are $\binom{8}{3} = 56$ possibilities with exactly three heads, of which 8 have three adjacent heads and $8 \cdot 4$ have exactly two adjacent heads (8 possibilities to place the two adjacent heads and 4 possibilities to place the third head). Finally, there are 2 possibilities using exactly four heads where no two of them are adjacent (heads and tails must alternate). There cannot be more than four heads without two of them being adjacent. Therefore there are $1 + 8 + (28 - 8) + (56 - 8 - 32) + 2 = 47$ possibilities with no adjacent heads, and the probability is $\frac{47}{256}$.

2018B 22. **Answer (C):** The set of all possible ordered pairs (x, y) occupies the unit square $0 \le x \le 1$, $0 \le y \le 1$ in the Cartesian plane. The numbers x, y, and 1 are the side lengths of a triangle if and only if $x + y > 1$, which means that (x, y) lies above the line $y = 1 - x$. By a generalization of the Pythagorean Theorem, the triangle is obtuse if and only if, in addition, $x^2 + y^2 < 1^2$, which means that (x, y) lies inside the circle of radius 1 centered at the origin. Within the unit square, the region inside the circle of radius 1 centered at the origin has area $\frac{\pi}{4}$, and the region below the line $y = 1 - x$ has area $\frac{1}{2}$. Therefore the ordered pairs that meet the required conditions occupy a region with area $\frac{\pi}{4} - \frac{1}{2} = \frac{\pi-2}{4}$. The area of the unit square is 1, so the required probability is also $\frac{\pi-2}{4} \approx \frac{1.14}{4} = 0.285$, which is closest to 0.29.



2001 23. **(D)** Think of continuing the drawing until all five chips are removed form the box. There are ten possible orderings of the colors: RRRWW, RRWRW, RWRRW, WRRRW, RRWWR, RWRWR, WRRWR, RWWRR, WRWRR, and WWRRR. The six orderings that end in R represent drawings that would have ended when the second white chip was drawn.

OR

Imagine drawing until only one chip remains. If the remaining chip is red, then that draw would have ended when the second white chip was removed. The last chip will be red with probability $3/5$.

2009B 23. **Answer (C):** After 10 min. = 600 sec., Rachel will have completed 6 laps and be 30 seconds from the finish line. Because Rachel runs one-fourth of a lap in 22.5 seconds, she will be in the picture taking region between

$$30 - \frac{22.5}{2} = 18.75 \quad \text{and} \quad 30 + \frac{22.5}{2} = 41.25$$

seconds of the 10th minute. After 10 minutes Robert will have completed 7 laps and will be 40 seconds from the starting line. Because Robert runs one-fourth of a lap in 20 seconds, he will be in the picture taking region between 30 and 50 seconds of the 10th minute. Hence both Rachel and Robert will be in the picture if it is taken between 30 and 41.25 seconds of the 10th minute. The probability that the picture is snapped during this time is

$$\frac{41.25 - 30}{60} = \frac{3}{16}.$$

2010A

23. **Answer (A):** If Isabella reaches the k^{th} box, she will draw a white marble from it with probability $\frac{k}{k+1}$. For $n \geq 2$, the probability that she will draw white marbles from each of the first $n-1$ boxes is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n},$$

so the probability that she will draw her first red marble from the n^{th} box is $P(n) = \frac{1}{n(n+1)}$. The condition $P(n) < 1/2010$ is equivalent to $n^2+n-2010 > 0$, from which $n > \frac{1}{2}(-1+\sqrt{8041})$ and $(2n+1)^2 > 8041$. The smallest positive odd integer whose square exceeds 8041 is 91, and the corresponding value of n is 45.

2002A

24. **(A)** There are ten ways for Tina to select a pair of numbers. The sums 9, 8, 4, and 3 can be obtained in just one way, and the sums 7, 6, and 5 can each be obtained in two ways. The probability for each of Sergio's choices is 1/10. Considering his selections in decreasing order, the total probability of Sergio's choice being greater is

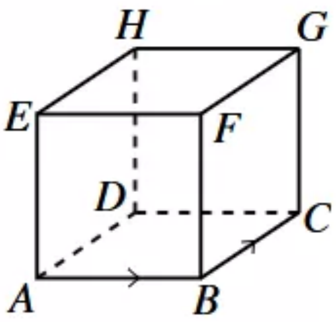
$$\left(\frac{1}{10}\right)\left(1+\frac{9}{10}+\frac{8}{10}+\frac{6}{10}+\frac{4}{10}+\frac{2}{10}+\frac{1}{10}+0+0+0\right) = \frac{2}{5}.$$

2009A

24. **Answer (C):** A plane that intersects at least three vertices of a cube either cuts into the cube or is coplanar with a cube face. Therefore the three randomly chosen vertices result in a plane that does not contain points inside the cube if and only if the three vertices come from the same face of the cube. There are 6 cube faces, so the number of ways to choose three vertices on the same cube face is $6 \cdot \binom{4}{3} = 24$. The total number of ways to choose the distinct vertices without restriction is $\binom{8}{3} = 56$. Hence the probability is $1 - \frac{24}{56} = \frac{4}{7}$.

2006A

25. **(C)** At each vertex there are three possible locations that the bug can travel to in the next move, so the probability that the bug will visit three different vertices after two moves is $2/3$. Label the first three vertices that the bug visits as A to B to C, as shown in the diagram. In order to visit every vertex, the bug must travel from C to either G or D.



The bug travels to G with probability $1/3$, and from there the bug must visit the vertices F, E, H, D in that order. Each of these choices has probability $1/3$ of occurring. So the probability that the path continues in the form

$$C \to G \to F \to E \to H \to D$$

is $\left(\frac{1}{3}\right)^5$.

Alternatively, the bug could travel from C to D with probability $1/3$, and then travel to H, which also occurs with probability $1/3$. From H the bug could go either to G or to E, with probability $2/3$, and from there to the two remaining vertices, each with probability $1/3$. So the probability that the path continues in one of the forms

$$C \to D \to H \begin{array}{l} \nearrow E \to F \to G \\ \searrow G \to F \to E \end{array}$$

is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$.

Hence the bug will visit every vertex in seven moves with probability

$$\left(\frac{2}{3}\right)\left[\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4\right] = \left(\frac{2}{3}\right)\left(\frac{1}{3} + \frac{2}{3}\right)\left(\frac{1}{3}\right)^4 = \frac{2}{243}.$$

OR

From a given starting point there are 3^7 possible walks of seven moves for the bug, all of them equally likely. If such a walk visits every vertex exactly once, there are three choices for the first move and, excluding a return to the start, two choices for the second. Label the first three vertices visited as $A, B,$ and C, in that order, and label the other vertices as shown. The bug must go to either G or D on its third move. In the first case it must then visit vertices $F, E, H,$ and D in order. In the second case it must visit either $H, E, F,$ and G or $H, G, F,$ and E in order. Thus there are $3 \cdot 2 \cdot 3 = 18$ walks that visit every vertex exactly once, so the required probability is $18/3^7 = 2/243$

2012A

25. **Answer (D):** It may be assumed that $x \leq y \leq z$. Because there are six possible ways of permuting the triple (x, y, z), it follows that the set of all triples (x, y, z) with $0 \leq x \leq y \leq z \leq n$ is a region whose volume is $\frac{1}{6}$ of the volume of the cube $[0, n]^3$, that is $\frac{1}{6}n^3$. Let S be the set of triples meeting the required condition. For every $(x, y, z) \in S$ consider the translation $(x, y, z) \mapsto (x', y', z') = (x, y - 1, z - 2)$. Note that $y' = y - 1 > x = x'$ and $z' = z - 2 > y - 1 = y'$. Thus the image of S under this translation is equal to $\{(x', y', z') : 0 \leq x' < y' < z' \leq n - 2\}$. Again by symmetry of the possible permutations of the triples (x', y', z'), the volume of this set is $\frac{1}{6}(n-2)^3$. Because $\frac{7^3}{9^3} = \frac{343}{729} < \frac{1}{2}$ and $\frac{8^3}{10^3} = \frac{512}{1000} > \frac{1}{2}$, the smallest possible value of n is 10.

2014B 25. **Answer (C):** First note that once the frog is on pad 5, it has probability $\frac{1}{2}$ of eventually being eaten by the snake, and a probability $\frac{1}{2}$ of eventually exiting the pond without being eaten. It is therefore necessary only to determine the probability that the frog on pad 1 will reach pad 5 before being eaten.

Consider the frog's jumps in pairs. The frog on pad 1 will advance to pad 3 with probability $\frac{9}{10} \cdot \frac{8}{10} = \frac{72}{100}$, will be back at pad 1 with probability $\frac{9}{10} \cdot \frac{2}{10} = \frac{18}{100}$, and will retreat to pad 0 and be eaten with probability $\frac{1}{10}$. Because the frog will eventually make it to pad 3 or make it to pad 0, the probability that it ultimately makes it to pad 3 is $\frac{72}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{36}{41}$, and the probability that it ultimately makes it to pad 0 is $\frac{10}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{5}{41}$.

Similarly, in a pair of jumps the frog will advance from pad 3 to pad 5 with probability $\frac{7}{10} \cdot \frac{6}{10} = \frac{42}{100}$, will be back at pad 3 with probability $\frac{7}{10} \cdot \frac{4}{10} + \frac{3}{10} \cdot \frac{8}{10} = \frac{52}{100}$, and will retreat to pad 1 with probability $\frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100}$. Because the frog will ultimately make it to pad 5 or pad 1 from pad 3, the probability that it ultimately makes it to pad 5 is $\frac{42}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{7}{8}$, and the probability that it ultimately makes it to pad 1 is $\frac{6}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{1}{8}$.

The sequences of pairs of moves by which the frog will advance to pad 5 without being eaten are

$$1 \to 3 \to 5, 1 \to 3 \to 1 \to 3 \to 5, 1 \to 3 \to 1 \to 3 \to 1 \to 3 \to 5,$$

and so on. The sum of the respective probabilities of reaching pad 5 is then

$$\begin{aligned}
&\frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \cdots \\
&= \frac{63}{82}\left(1 + \frac{9}{82} + \left(\frac{9}{82}\right)^2 + \cdots\right) \\
&= \frac{63}{82} \div \left(1 - \frac{9}{82}\right) \\
&= \frac{63}{73}.
\end{aligned}$$

Therefore the requested probability is $\frac{1}{2} \cdot \frac{63}{73} = \frac{63}{146}$.

OR

For $1 \le j \le 5$, let p_j be the probability that the frog eventually reaches pad 10 starting at pad j. By symmetry $p_5 = \frac{1}{2}$. For the frog to reach pad 10 starting from pad 4, the frog goes either to pad 3 with probability $\frac{2}{5}$ or to pad 5 with probability $\frac{3}{5}$, and then continues on a successful sequence from either of these pads. Thus $p_4 = \frac{2}{5}p_3 + \frac{3}{5}p_5 = \frac{2}{5}p_3 + \frac{3}{10}$. Similarly, to reach pad 10 starting from pad 3, the frog goes either to pad 2 with probability $\frac{3}{10}$ or to pad 4 with probability $\frac{7}{10}$. Thus $p_3 = \frac{3}{10}p_2 + \frac{7}{10}p_4$, and substituting from the previous equation for p_4 gives $p_3 = \frac{5}{12}p_2 + \frac{7}{24}$. In the same way, $p_2 = \frac{1}{5}p_1 + \frac{4}{5}p_3$ and after substituting for p_3 gives $p_2 = \frac{3}{10}p_1 + \frac{7}{20}$. Lastly, for the frog to escape starting from pad 1, it is necessary for it to get to pad 2 with probability $\frac{9}{10}$, and then escape starting from pad 2. Thus $p_1 = \frac{9}{10}p_2 = \frac{9}{10}\left(\frac{3}{10}p_1 + \frac{7}{20}\right)$, and solving the equation gives $p_1 = \frac{63}{146}$.

Note: This type of random process is called a Markov process.

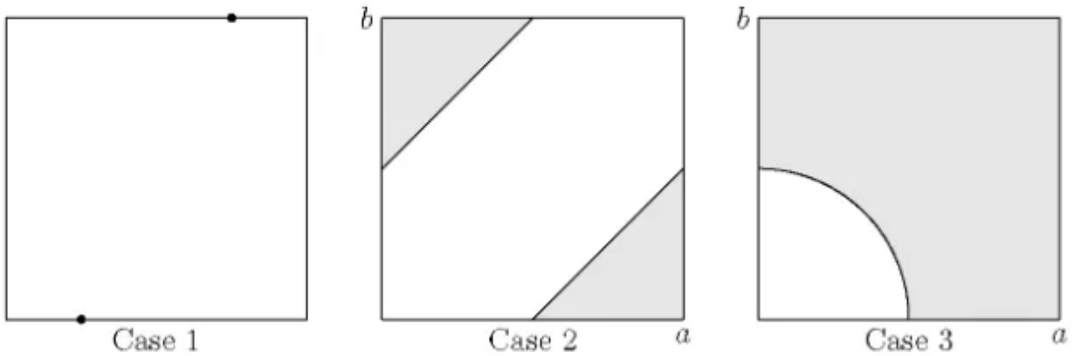
2015A

25. **Answer (A):** Let the square have vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, and consider three cases.

Case 1: The chosen points are on opposite sides of the square. In this case the distance between the points is at least $\frac{1}{2}$ with probability 1.

Case 2: The chosen points are on the same side of the square. It may be assumed that the points are $(a,0)$ and $(b,0)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy either $b \ge a + \frac{1}{2}$ or $b \le a - \frac{1}{2}$. These inequalities describe the union of two isosceles right triangles with leg length $\frac{1}{2}$, together with their interiors. The area of the region is $\frac{1}{4}$, and the area of the square is 1, so the probability that the pair of points meets the requirement in this case is $\frac{1}{4}$.

Case 3: The chosen points are on adjacent sides of the square. It may be assumed that the points are $(a,0)$ and $(0,b)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy $\sqrt{a^2+b^2} \ge \frac{1}{2}$. These inequalities describe the region inside the square and outside a quarter-circle of radius $\frac{1}{2}$. The area of this region is $1-\frac{1}{4}\pi(\frac{1}{2})^2 = 1-\frac{\pi}{16}$, which is also the probability that the pair of points meets the requirement in this case.



Case 1 Case 2 Case 3

Cases 1 and 2 each occur with probability $\frac{1}{4}$, and Case 3 occurs with probability $\frac{1}{2}$. The requested probability is

$$\frac{1}{4}\cdot 1 + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{2}\left(1-\frac{\pi}{16}\right) = \frac{26-\pi}{32},$$

and $a+b+c = 59$.