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DIVISOR/GCF

2009B

21. **Answer (D):** The sum of any four consecutive powers of 3 is divisible by $3^0 + 3^1 + 3^2 + 3^3 = 40$ and hence is divisible by 8. Therefore

$$(3^2 + 3^3 + 3^4 + 3^5) + \cdots + (3^{2006} + 3^{2007} + 3^{2008} + 3^{2009})$$

is divisible by 8. So the required remainder is $3^0 + 3^1 = 4$.

2005B

22. **(C)** Since

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

the condition is equivalent to having an integer value for

$$\frac{n!}{n(n+1)/2}.$$

This reduces, when $n \geq 1$, to having an integer value for

$$\frac{2(n-1)!}{n+1}.$$

This fraction is an integer unless $n+1$ is an odd prime. There are 8 odd primes less than or equal to 25, so there are $24 - 8 = 16$ numbers less than or equal to 24 that satisfy the condition.

2016A

2018A

22. **Answer (D):** Because $\gcd(a,b) = 24 = 2^3 \cdot 3$ and $\gcd(b,c) = 36 = 2^2 \cdot 3^2$, it follows that a is divisible by 2 and 3 but not by 3^2. Similarly, because $\gcd(b,c) = 2^2 \cdot 3^2$ and $\gcd(c,d) = 54 = 2 \cdot 3^3$, it follows that d is divisible by 2 and 3 but not by 2^2. Therefore $\gcd(d,a) = 2 \cdot 3 \cdot n$, where n is a product of primes that do not include 2 or 3. Because $70 < \gcd(d,a) < 100$ and n is an integer, it must be that $12 \leq n \leq 16$, so $n = 13$, and 13 must also be a divisor of a. The conditions are satisfied if $a = 2^3 \cdot 3 \cdot 13 = 312$, $b = 2^3 \cdot 3^2 = 72$, $c = 2^2 \cdot 3^3 = 108$, and $d = 2 \cdot 3^3 \cdot 13 = 702$.

2011B

23. **Answer (D):** In the expansion of $(2000+11)^{2011}$, all terms except 11^{2011} are divisible by 1000, so the hundreds digit of 2011^{2011} is equal to that of 11^{2011}. Furthermore, in the expansion of $(10+1)^{2011}$, all terms except 1^{2011}, $\binom{2011}{1}(10)\left(1^{2010}\right)$, and $\binom{2011}{2}(10)^2\left(1^{2009}\right)$ are divisible by 1000. Thus the hundreds digit of 2011^{2011} is equal to that of

$$\begin{aligned} & 1+\binom{2011}{1}(10)\left(1^{2010}\right)+\binom{2011}{2}(10)^2\left(1^{2009}\right) \\ = & 1+2011\cdot 10+2011\cdot 1005\cdot 100 \\ = & 1+2011\cdot 100510. \end{aligned}$$

Finally, the hundreds digit of this number is equal to that of $1+11\cdot 510=5611$, so the requested hundreds digit is 6.

2017B

23. **Answer (C):** The remainder when N is divided by 5 is clearly 4. A positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits of N is $4(0+1+2+\cdots+9)+10\cdot 1+10\cdot 2+10\cdot 3+(4+0)+(4+1)+(4+2)+(4+3)+(4+4)=270$, so N must be a multiple of 9. Then $N-9$ must also be a multiple of 9, and the last digit of $N-9$ is 5, so it is also a multiple of 5. Thus $N-9$ is a multiple of 45, and N leaves a remainder of 9 when divided by 45.

2018B

23. **Answer (B):** Recall that $a \cdot b = \gcd(a,b) \cdot \operatorname{lcm}(a,b)$. Let $x = \operatorname{lcm}(a,b)$ and $y = \gcd(a,b)$. The given equation is then $xy + 63 = 20x + 12y$, which can be rewritten as

$$(x-12)(y-20) = 240 - 63 = 177 = 3 \cdot 59 = 1 \cdot 177.$$

Because x and y are integers, one of the following must be true:

- $x - 12 = 1$ and $y - 20 = 177$,
- $x - 12 = 177$ and $y - 20 = 1$,
- $x - 12 = 3$ and $y - 20 = 59$,
- $x - 12 = 59$ and $y - 20 = 3$.

Therefore (x,y) must be $(13,197)$, $(189,21)$, $(15,79)$, or $(71,23)$. Because x must be a multiple of y, only $(x,y) = (189,21)$ is possible. Therefore $\gcd(a,b) = 21 = 7 \cdot 3$, and $\operatorname{lcm}(a,b) = 189 = 7 \cdot 3^3$. Both a and b are divisible by 7 but not by 7^2; one of a and b is divisible by 3 but not 3^2, and the other is divisible by 3^3 but not 3^4; and neither is divisible by any other prime. Therefore one of them is $7 \cdot 3 = 21$ and the other is $7 \cdot 3^3 = 189$. There are 2 ordered pairs, $(a,b) = (21,189)$ and $(a,b) = (189,21)$.

2008A

24. **Answer (D):** The units digit of 2^n is 2, 4, 8, and 6 for $n = 1$, 2, 3, and 4, respectively. For $n > 4$, the units digit of 2^n is equal to that of 2^{n-4}. Thus for every positive integer j the units digit of 2^{4j} is 6, and hence 2^{2008} has a units digit of 6. The units digit of 2008^2 is 4. Therefore the units digit of k is 0, so the units digit of k^2 is also 0. Because 2008 is even, both 2008^2 and 2^{2008} are multiples of 4. Therefore k is a multiple of 4, so the units digit of 2^k is 6, and the units digit of $k^2 + 2^k$ is also 6.

2010A

24. **Answer (A):** There are 18 factors of 90! that are multiples of 5, 3 factors that are multiples of 25, and no factors that are multiples of higher powers of 5. Also, there are more than 45 factors of 2 in 90!. Thus $90! = 10^{21}N$ where N is an integer not divisible by 10, and if $N \equiv n \pmod{100}$ with $0 < n \leq 99$, then n is a multiple of 4.

Let $90! = AB$ where A consists of the factors that are relatively prime to 5 and B consists of the factors that are divisible by 5. Note that $\prod_{j=1}^{4}(5k+j) \equiv 5k(1+2+3+4)+1\cdot 2\cdot 3\cdot 4 \equiv 24 \pmod{25}$, thus

$$\begin{aligned} A &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot\cdots\cdot(86\cdot 87\cdot 88\cdot 89) \\ &\equiv 24^{18} \equiv (-1)^{18} \equiv 1 \pmod{25}. \end{aligned}$$

Similarly,

$$B = (5\cdot 10\cdot 15\cdot 20)\cdot(30\cdot 35\cdot 40\cdot 45)\cdot(55\cdot 60\cdot 65\cdot 70)\cdot(80\cdot 85\cdot 90)\cdot(25\cdot 50\cdot 75),$$

thus

$$\begin{aligned} \frac{B}{5^{21}} &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot(11\cdot 12\cdot 13\cdot 14)\cdot(16\cdot 17\cdot 18)\cdot(1\cdot 2\cdot 3) \\ &\equiv 24^3\cdot(-9)\cdot(-8)\cdot(-7)\cdot 6 \equiv (-1)^3\cdot 1 \equiv -1 \pmod{25}. \end{aligned}$$

Finally, $2^{21} = 2\cdot(2^{10})^2 = 2\cdot(1024)^2 \equiv 2\cdot(-1)^2 \equiv 2 \pmod{25}$, so $13\cdot 2^{21} \equiv 13\cdot 2 \equiv 1 \pmod{25}$. Therefore

$$\begin{aligned} N &\equiv (13\cdot 2^{21})N = 13\cdot\frac{90!}{5^{21}} = 13\cdot A\cdot\frac{B}{5^{21}} \equiv 13\cdot 1\cdot(-1) \pmod{25} \\ &\equiv -13 \equiv 12 \pmod{25}. \end{aligned}$$

Thus n is equal to 12, 37, 62, or 87, and because n is a multiple of 4, it follows that $n = 12$.

2013B 24. **Answer (A):** Let n denote a nice number from the given set. An integer m has exactly four divisors if and only if $m = p^3$ or $m = pq$, where p and q (with $p > q$) are prime numbers. In the former case, the sum of the four divisor is equal to $1 + p + p^2 + p^3$. Note that $1 + 11 + 11^2 + 11^3 < 2010 \le n$ and $1 + 13 + 13^2 + 13^3 > 2019 \ge n$. Therefore we must have $m = pq$ and $n = 1+q+p+pq = (1+q)(1+p)$. Because p is odd, n must be an even number. If $q = 2$, then n must be divisible by 3. In the given set only $2010 = (1+2)(1+669)$ and $2016 = (1 + 2)(1 + 671)$ satisfy these requirements. However neither 669 nor 671 are prime. If q is odd, then n must be divisible by 4. In the given set, only 2012 and 2016 are divisible by 4. None of the pairs of factors of 2012, namely $1 \cdot 2012$, $2 \cdot 1006$, $4 \cdot 503$, gives rise to primes p and q. This leaves $2016 = (1 + 3)(1 + 503)$, which is the only nice number in the given set.

Remark: Note that 2016 is nice in five ways. The other four ways are $(1+7)(1+251)$, $(1 + 11)(1 + 167)$, $(1 + 23)(1 + 83)$, and $(1 + 41)(1 + 47)$.

2014A 24. **Answer (A):** After the nth iteration there will be $4+5+6+\cdots+(n+3) = \frac{(n+3)(n+4)}{2} - 6 = \frac{n(n+7)}{2}$ numbers listed, and $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ numbers skipped. The first number to be listed on the $(n + 1)$st iteration will be one more than the sum of these, or $n^2 + 4n + 1$.

It is necessary to find the greatest integer value of n such that $\frac{n(n+7)}{2} < 500{,}000$. This implies that $n(n + 7) < 1{,}000{,}000$. Note that, for $n = 993$, this product becomes $993 \cdot 1000 = 993{,}000$. Next observe that, in general, $(a + k)(b + k) = ab + (a+b)k + k^2$ so $(993+k)(1000+k) = 993{,}000+1993k+k^2$. By inspection, the largest integer value of k that will satisfy the above inequality is 3 and the n needed is 996. After the 996th iteration, there will be $\frac{993{,}000+1993\cdot 3+9}{2} = \frac{998{,}988}{2} = 499{,}494$ numbers in the sequence. The 997th iteration will begin with the number $996^2 + 4 \cdot 996 + 1 = 996 \cdot 1000 + 1 = 996{,}001$.

The 506th number in the 997th iteration will be the 500,000th number in the sequence. This is $996{,}001 + 505 = 996{,}506$.

2001

25. **(B)** For integers not exceeding 2001, there are $\lfloor 2001/3 \rfloor = 667$ multiples of 3 and $\lfloor 2001/4 \rfloor = 500$ multiples of 4. The total, 1167, counts the $\lfloor 2001/12 \rfloor = 166$ multiples of 12 twice, so there are $1167 - 166 = 1001$ multiples of 3 or 4. From these we exclude the $\lfloor 2001/15 \rfloor = 133$ multiples of 15 and the $\lfloor 2001/20 \rfloor = 100$ multiples of 20, since these are multiples of 5. However, this excludes the $\lfloor 2001/60 \rfloor = 33$ multiples of 60 twice, so we must re-include these. The number of integers satisfying the conditions is $1001 - 133 - 100 + 33 = 801$.

2003B

25. (B) A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So a four-digit number $ab23$ is divisible by 3 if and only if the two-digit number ab leaves a remainder of 1 when divided by 3. There are 90 two-digit numbers, of which $90/3 = 30$ leave a remainder of 1 when divided by 3.

2007B

25. **Answer (A):** Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that

$$u + \frac{14}{9u} = k$$

for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

Rewrite the equation

$$\frac{a}{b} + \frac{14b}{9a} = k$$

in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

2016A

21. **Answer (D):** Let X be the foot of the perpendicular from P to $\overline{QQ'}$, and let Y be the foot of the perpendicular from Q to $\overline{RR'}$. By the Pythagorean Theorem,

$$P'Q' = PX = \sqrt{(2+1)^2 - (2-1)^2} = \sqrt{8}$$

and

$$Q'R' = QY = \sqrt{(3+2)^2 - (3-2)^2} = \sqrt{24}.$$

The required area can be computed as the sum of the areas of the two smaller trapezoids, $PQQ'P'$ and $QRR'Q'$, minus the area of the large trapezoid, $PRR'P'$:

$$\frac{1+2}{2}\sqrt{8} + \frac{2+3}{2}\sqrt{24} - \frac{1+3}{2}\left(\sqrt{8}+\sqrt{24}\right) = \sqrt{6} - \sqrt{2}.$$