12

SOLVE FOR X

2001

22. **(D)** Since v appears in the first row, first column, and on diagonal, the sum of the remaining two numbers in each of these lines must be the same. Thus,

$$25 + 18 = 24 + w = 21 + x,$$

so $w = 19$ and $x = 22$. now 25,22, and 19 form a diagonal with a sum of 66, so we can find $v = 23$, $y = 26$, and $z = 20$. Hence $y + z = 46$.

2005A 21. **(B)** Because

$$1+2+\cdots+n=\frac{n(n+1)}{2},$$

$1+2+\cdots+n$ divides the positive integer $6n$ if and only if

$$\frac{6n}{n(n+1)/2}=\frac{12}{n+1} \text{ is an integer.}$$

There are five such positive values of n, namely, 1, 2, 3, 5, and 11.

2005A 22. **(D)** The sets S and T consist, respectively, of the positive multiples of 4 that do not exceed $2005\cdot 4=8020$ and the positive multiples of 6 that do not exceed $2005\cdot 6=12{,}030$. Thus $S\bigcap T$, the set of numbers that are common to S and to T, consists of the positive multiples of 12 that do not exceed 8020. Let $\lfloor x\rfloor$ represent the largest integer that is less than or equal to x. Then the number of elements in the set $S\bigcap T$ is

$$\left\lfloor\frac{8020}{12}\right\rfloor=\left\lfloor 668+\frac{1}{3}\right\rfloor=668.$$

2007A 23. **Answer (B):** Let x and y be, respectively, the larger and smaller of the integers. Then $96=x^2-y^2=(x+y)(x-y)$. Because 96 is even, x and y are both even or are both odd. In either case $x+y$ and $x-y$ are both even. Hence there are four possibilities for $(x+y,x-y)$, which are $(48,2)$, $(24,4)$, $(16,6)$, and $(12,8)$. The four corresponding values of (x,y) are $(25,23)$, $(14,10)$, $(11,5)$, and $(10,2)$.

2015B

23. **Answer (B):** Because there are ample factors of 2, it is enough to count the number of factors of 5. Let $f(n)$ be the number of factors of 5 in positive integers less than or equal to n. For n from 5 to 9, $f(n) = 1$. In order for $f(2n)$ to equal 3, $2n$ must be between 15 and 19, inclusive. Therefore $n = 8$ or $n = 9$. For n from 10 to 14, $f(n) = 2$. In order for $f(2n)$ to equal 6, $2n$ must be between 25 and 29, inclusive. Hence, $n = 13$ or $n = 14$. Thus the four smallest integers n that satisfy the specified condition are 8, 9, 13, and 14. Their sum is 44 and the sum of the digits of 44 is 8.

OR

In fact there are only 4 possible values of n. By Legendre's Theorem, if $n!$ ends in k zeros and $(2n)!$ ends in k' zeros, then

$$k = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^j} \right\rfloor,$$
$$k' = \left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{5^2} \right\rfloor + \left\lfloor \frac{2n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{2n}{5^j} \right\rfloor + \left\lfloor \frac{2n}{5^{j+1}} \right\rfloor,$$

where j is the highest power of 5 not exceeding n, and thus the highest power of 5 not exceeding $2n$ is at most $j+1$. If x is a real number, then $\lfloor 2x \rfloor \le 2\lfloor x \rfloor + 1$. So $\left\lfloor \frac{2n}{5^i} \right\rfloor \le 2\left\lfloor \frac{n}{5^i} \right\rfloor + 1$ for each $1 \le i \le j+1$. Adding these inequalities yields $k' \le 2k + j + 1$. If $n \ge 15$, then $k > 2 + j - 1 = j + 1$ and $k' < 3k$. For $n = 13$ and $n = 14$, $k = 2$ and $k' = 5 + 1 = 6 = 3k$. For $n \le 12$, $k = \left\lfloor \frac{n}{5} \right\rfloor$ and $k' = \left\lfloor \frac{2n}{5} \right\rfloor$; in this case $k' = 3k$ only for $n = 8$ and $n = 9$. So $s = 8 + 9 + 13 + 14 = 44$ and the answer is $4 + 4 = 8$.

2005A

24. **(B)** The conditions imply that both n and $n + 48$ are squares of primes. So for each successful value of n we have primes p and q with $p^2 = n + 48$ and $q^2 = n$, and

$$48 = p^2 - q^2 = (p+q)(p-q).$$

The pairs of factors of 48 are

48 and 1, 24 and 2, 16 and 3, 12 and 4, and 8 and 6.

These give pairs (p, q), respectively, of

$$\left(\frac{49}{2}, \frac{47}{2}\right), \quad (13, 11), \quad \left(\frac{19}{2}, \frac{13}{2}\right), \quad (8, 4), \quad \text{and} \quad (7, 1).$$

Only $(p, q) = (13, 11)$ gives prime values for p and for q, with $n = 11^2 = 121$ and $n + 48 = 13^2 = 169$.

2005B 24. **(E)** By the given conditions, it follows that $x > y$. Let $x = 10a + b$ and $y = 10b + a$, where $a > b$. Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since $99(a^2 - b^2)$ must be a perfect square,

$$a^2 - b^2 = (a + b)(a - b) = 11k^2,$$

for some positive integer k. Because a and b are distinct digits, we have $a - b \leq 9 - 1 = 8$ and $a + b \leq 9 + 8 = 17$. It follows that $a + b = 11$, $a - b = k^2$, and k is either 1 or 2.

If $k = 2$, then $(a, b) = (15/2, 7/2)$, which is impossible. Thus $k = 1$ and $(a, b) = (6, 5)$. This gives $x = 65$, $y = 56$, $m = 33$, and $x + y + m = 154$.

2012A 24. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a - b$, $b - c$, and $c - a$ are integers with their sum equal to 0 and $a \geq b \geq c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a, b, c) = (c + 3, c + 1, c)$ or $(a, b, c) = (c + 3, c + 2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a - c)(a + c) + (a - b)b = 3(2c + 3) + 2(c + 1)$. Solving gives $(a, b, c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

2014A

25. **Answer (B):** Because $2^2 < 5$ and $2^3 > 5$, there are either two or three integer powers of 2 strictly between any two consecutive integer powers of 5. Thus for each n there is at most one m satisfying the given inequalities, and the question asks for the number of cases in which there are three powers rather than two. Let d (respectively, t) be the number of nonnegative integers n less than 867 such that there are exactly two (respectively, three) powers of 2 strictly between 5^n and 5^{n+1}. Because $2^{2013} < 5^{867} < 2^{2014}$, it follows that $d + t = 867$ and $2d + 3t = 2013$. Solving the system yields $t = 279$.