14

ALGEBRA WORD PROBLEMS

2015B

21. **Answer (D):** Assume that there are t steps in this staircase and it took Dash $d+1$ jumps. Then the possible values of t are $5d+1, 5d+2, 5d+3, 5d+4, 5d+5$. On the other hand, it took Cozy $d+20$ jumps, and $t = 2d+39$ or $t = 2d+40$.

There are 10 possible combinations but only 3 of them lead to integer values of d: $t = 5d+3 = 2d+39$, or $t = 5d+1 = 2d+40$, or $t = 5d+4 = 2d+40$. The possible values of t are 63, 66, and 64, and $s = 63+66+64 = 193$. The answer is $1+9+3 = 13$.

2018B

21. **Answer (C):** Let d be the next divisor of n after 323. Then $\gcd(d, 323) \neq 1$, because otherwise $n \geq 323d > 323^2 > 100^2 = 10000$, contrary to the fact that n is a 4-digit number. Therefore $d - 323 \geq \gcd(d, 323) > 1$. The prime factorization of 323 is $17 \cdot 19$. Thus the next divisor of n is at least $323 + 17 = 340 = 17 \cdot 20$. Indeed, 340 will be the next number in Mary's list when $n = 17 \cdot 19 \cdot 20 = 6460$.

2002A

22. **(C)** The first application removes ten tiles, leaving 90. The second and third applications each remove nine tiles leaving 81 and 72, respectively. Following this pattern, we consecutively remove $10, 9, 9, 8, 8, \ldots, 2, 2, 1$ tiles before we are left with only one. This requires $1 + 2(8) + 1 = 18$ applications.

OR

Starting with n^2 tiles, the first application leaves $n^2 - n$ tiles. The second application reduces the number to $n^2 - n - (n-1) = (n-1)^2$ tiles. Since two applications reduce the number from n^2 to $(n-1)^2$, it follows that $2(n-1)$ applications reduce the number from n^2 to $(n-(n-1))^2 = 1$, and $2(10-1) = 18$.

2006A

22. **(C)** If a debt of D dollars can be resolved in this way, then integers p and g must exist with

$$D = 300p + 210g = 30(10p + 7g).$$

As a consequence, D must be a multiple of 30, and no positive debt less than \$30 can be resolved. A debt of \$30 can be resolved since

$$30 = 300(-2) + 210(3).$$

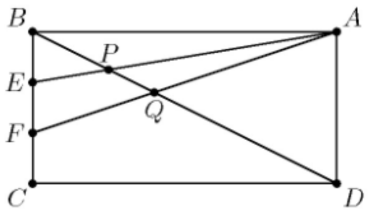
This is done by giving 3 goats and receiving 2 pigs.

2006B

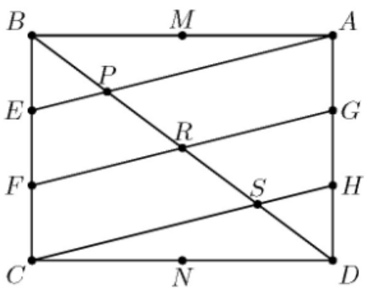
22. **(D)** The total cost of the peanut butter and jam is $N(4B+5J) = 253$ cents, so N and $4B+5J$ are factors of $253 = 11 \cdot 23$. Because $N > 1$, the possible values of N are 11, 23, and 253. If $N = 253$, then $4B + 5J = 1$, which is impossible since B and J are positive integers. If $N = 23$, then $4B + 5J = 11$, which also has no solutions in positive integers. Hence $N = 11$ and $4B + 5J = 23$, which has the unique positive integer solution $B = 2$ and $J = 3$. So the cost of the jam is $11(3)(5¢) = \$1.65$.

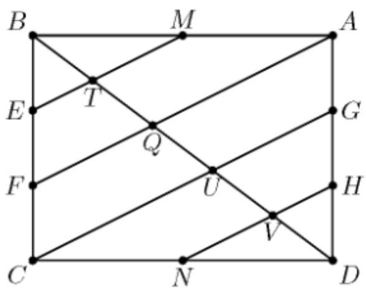
2016A

19. **Answer (E):** Triangles APD and EPB are similar and $BE : DA = 1 : 3$, so $BP = \frac{1}{4}BD$. Triangles AQD and FQB are similar and $BF : DA = 2 : 3$, so $BQ = \frac{2}{5}BD$ and $QD = \frac{3}{5}BD$. Then $PQ = BQ - BP = (\frac{2}{5} - \frac{1}{4})BD = \frac{3}{20}BD$. Thus $BP : PQ : QD = \frac{1}{4} : \frac{3}{20} : \frac{3}{5} = 5 : 3 : 12$, and $r + s + t = 5 + 3 + 12 = 20$.



Note: The answer is independent of the dimensions of the original rectangle. Consider the figures below, showing the rectangle $ABCD$ with points E and F trisecting side $\overline{BC}$. Let G and H trisect $\overline{AD}$, and let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$. Then the segments $\overline{AE}$, $\overline{GF}$, and $\overline{HC}$ are equally spaced, implying that $BP = PR = RS = SD$ and showing that $BP : PD : BD = 1 : 3 : 4 = 5 : 15 : 20$. The segments $\overline{ME}$, $\overline{AF}$, $\overline{GC}$, and $\overline{HN}$ are also equally spaced, implying that $BT = TQ = QU = UV = VD$ and showing that $BQ : QD : BD = 2 : 3 : 5 = 8 : 12 : 20$. It then follows that $BP : PQ : QD = 5 : (15 - 12) : 12 = 5 : 3 : 12$.





2007B

24. **Answer (C):** Since n is divisible by 9, the sum of the digits of n must be a multiple of 9. At least one digit of n is 4, so at least nine digits must be 4, and at least one digit must be 9. For n to be divisible by 4, the last two digits of n must each be 4. These conditions are satisfied by several ten-digit numbers, of which the smallest is 4,444,444,944.

2009A

25. **Answer (B):** Note that $I_k = 2^{k+2} \cdot 5^{k+2} + 2^6$. For $k < 4$, the first term is not divisible by 2^6, so $N(k) < 6$. For $k > 4$, the first term is divisible by 2^7, but the second term is not, so $N(k) < 7$. For $k = 4$, $I_4 = 2^6(5^6 + 1)$, and because the second factor is even, $N(4) \geq 7$. In fact the second factor is a sum of cubes so

$$(5^6 + 1) = ((5^2)^3 + 1^3) = (5^2 + 1)((5^2)^2 - 5^2 + 1).$$

The factor $5^2 + 1 = 26$ is divisible by 2 but not 4, and the second factor is odd, so $5^6 + 1$ contributes one more factor of 2. Hence the maximum value for $N(k)$ is 7.

2013B

25. **Answer (E):** Expand the set of three-digit positive integers to include integers N, $0 \le N \le 99$, with leading zeros appended. Because $\text{lcm}(5^2, 6^2, 10^2) = 900$, such an integer N meets the required condition if and only if $N + 900$ does. Therefore N can be considered to be chosen from the set of integers between 000 and 899, inclusive. Suppose that the last two digits in order of the base-5 representation of N are a_1 and a_0. Similarly, suppose that the last two digits of the base-6 representation of N are b_1 and b_0. By assumption, $2N \equiv a_0 + b_0 \pmod{10}$, but $N \equiv a_0 \pmod 5$ and so

$$a_0 + b_0 \equiv 2N \equiv 2a_0 \pmod{10}.$$

Thus $a_0 \equiv b_0 \pmod{10}$ and because $0 \le a_0 \le 4$ and $0 \le b_0 \le 5$, it follows that $a_0 = b_0$. Because $N \equiv a_0 \pmod 5$, it follows that there is an integer N_1 such that $N = 5N_1 + a_0$. Also, $N \equiv a_0 \pmod 6$ implies that $5N_1 + a_0 \equiv a_0 \pmod 6$ and so $N_1 \equiv 0 \pmod 6$. It follows that $N_1 = 6N_2$ for some integer N_2 and so $N = 30N_2 + a_0$. Similarly, $N \equiv 5a_1 + a_0 \pmod{25}$ implies that $30N_2 + a_0 \equiv 5a_1 + a_0 \pmod{25}$ and then $N_2 \equiv 6N_2 \equiv a_1 \pmod 5$. It follows that $N_2 = 5N_3 + a_1$ for some integer N_3 and so $N = 150N_3 + 30a_1 + a_0$. Once more, $N \equiv 6b_1 + a_0 \pmod{36}$ implies that $6N_3 - 6a_1 + a_0 \equiv 150N_3 + 30a_1 + a_0 \equiv 6b_1 + a_0 \pmod{36}$ and then $N_3 \equiv a_1 + b_1 \pmod 6$. It follows that $N_3 = 6N_4 + a_1 + b_1$ for some integer N_4 and so $N = 900N_4 + 180a_1 + 150b_1 + a_0$. Finally, $2N \equiv 10(a_1 + b_1) + 2a_0 \pmod{100}$ implies that

$$60a_1 + 2a_0 \equiv 360a_1 + 300b_1 + 2a_0 \equiv 10a_1 + 10b_1 + 2a_0 \pmod{100}.$$

Therefore $5a_1 \equiv b_1 \pmod{10}$, equivalently, $b_1 \equiv 0 \pmod 5$ and $a_1 \equiv b_1 \pmod 2$. Conversely, if $N = 900N_4 + 180a_1 + 150b_1 + a_0$, $a_0 = b_0$, and $5a_1 \equiv b_1 \pmod{10}$, then $2N \equiv 60a_1 + 2a_0 = 10(a_1 + 5a_1) + a_0 + b_0 \equiv 10(a_1 + b_1) + (a_0 + b_0) \pmod{100}$. Because $0 \le a_1 \le 4$ and $0 \le b_1 \le 5$, it follows that there are exactly 5 different pairs (a_1, b_1), namely $(0,0)$, $(2,0)$, $(4,0)$, $(1,5)$, and $(3,5)$. Each of these can be combined with 5 different values of a_0 ($0 \le a_0 \le 4$), to determine exactly 25 different numbers N with the required property.

2015B

25. **Answer (B):** Because the volume and surface area are numerically equal, $abc = 2(ab+ac+bc)$. Rewriting the equation as $ab(c-6)+ac(b-6)+bc(a-6) = 0$ shows that $a \leq 6$. The original equation can also be written as $(a-2)bc-2ab-2ac = 0$. Note that if $a = 2$, this becomes $b+c = 0$, and there are no solutions. Otherwise, multiplying both sides by $a-2$ and adding $4a^2$ to both sides gives $[(a-2)b-2a][(a-2)c-2a] = 4a^2$. Consider the possible values of a.

$a = 1$: $(b+2)(c+2) = 4$
There are no solutions in positive integers.

$a = 3$: $(b-6)(c-6) = 36$
The 5 solutions for (b, c) are $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, and $(12, 12)$.

$a = 4$: $(b-4)(c-4) = 16$
The 3 solutions for (b, c) are $(5, 20)$, $(6, 12)$, and $(8, 8)$.

$a = 5$: $(3b-10)(3c-10) = 100$
Each factor must be congruent to 2 modulo 3, so the possible pairs of factors are $(2, 50)$ and $(5, 20)$. The solutions for (b, c) are $(4, 20)$ and $(5, 10)$, but only $(5, 10)$ has $a \leq b$.

$a = 6$: $(b-3)(c-3) = 9$
The solutions for (b, c) are $(4, 12)$ and $(6, 6)$, but only $(6, 6)$ has $a \leq b$.

Thus in all there are 10 ordered triples (a, b, c): $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, and $(6, 6, 6)$.

2018A

25. **Answer (D):** The equation $C_n - B_n = A_n^2$ is equivalent to

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \left(\frac{10^n - 1}{9} \right)^2.$$

Dividing by $10^n - 1$ and clearing fractions yields

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

As this must hold for two different values n_1 and n_2, there are two such equations, and subtracting them gives

$$\left(9c - a^2\right)\left(10^{n_1} - 10^{n_2}\right) = 0.$$

The second factor is non-zero, so $9c - a^2 = 0$ and thus $9b - 9c - a^2 = 0$. From this it follows that $c = \left(\frac{a}{3}\right)^2$ and $b = 2c$. Hence digit a must be 3, 6, or 9, with corresponding values 1, 4, or 9 for c, and 2, 8, or

18 for b. The case $b = 18$ is invalid, so there are just two triples of possible values for a, b, and c, namely $(3, 2, 1)$ and $(6, 8, 4)$. In fact, in these cases, $C_n - B_n = A_n^2$ for *all* positive integers n; for example, $4444 - 88 = 4356 = 66^2$. The second triple has the greater coordinate sum, $6 + 8 + 4 = 18$.