16

SEQUENCE AND SERIES

2004B 21. **(A)** The smallest number that appears in both sequences is 16. The two sequences have common differences 3 and 7, whose least common multiple is 21, so a number appears in both sequences if and only if it is in the form

$$16 + 21k,$$

where k is a nonnegative integer. Such a number is in the first 2004 terms of both sequences if and only if

$$16 + 21k \leq 1 + 2003(3) = 6010.$$

Thus $0 \leq k \leq 285$, so there are 286 duplicate numbers. Therefore the number of distinct numbers is $4008 - 286 = 3722$.

2010B 24. **Answer (E):** The Raiders' score was $a(1+r+r^2+r^3)$, where a is a positive integer and $r > 1$. Because ar is also an integer, $r = m/n$ for relatively prime positive integers m and n with $m > n$. Moreover $ar^3 = a \cdot \frac{m^3}{n^3}$ is an integer, so n^3 divides a. Let $a = n^3 A$. Then the Raiders' score was $R = A(n^3 + mn^2 + m^2 n + m^3)$, and the Wildcats' score was $R-1 = a+(a+d)+(a+2d)+(a+3d) = 4a+6d$ for some positive integer d. Because $A \geq 1$, the condition $R \leq 100$ implies that $n \leq 2$ and $m \leq 4$. The only possibilities are $(m,n) = (4,1), (3,2), (3,1), (4,1)$, or $(2,1)$. The corresponding values of R are, respectively, $85A$, $65A$, $40A$, and $15A$. In the first two cases $A = 1$, and the corresponding values of $R-1$ are, respectively, $64 = 32+6d$ and $84 = 4+6d$. In neither case is d an integer. In the third case $40A = 40a = 4a + 6d + 1$ which is impossible in integers. In the last case $15a = 4a + 6d + 1$, from which $11a = 6d + 1$. The only solution in positive integers for which $4a+6d \leq 100$ is $(a,d) = (5,9)$. Thus $R = 5+10+20+40 = 75$, $R-1 = 5+14+23+32 = 74$, and the number of points scored in the first half was $5 + 10 + 5 + 14 = 34$.

2013B 24. **Answer (A):** Let n denote a nice number from the given set. An integer m has exactly four divisors if and only if $m = p^3$ or $m = pq$, where p and q (with $p > q$) are prime numbers. In the former case, the sum of the four divisor is equal to $1 + p + p^2 + p^3$. Note that $1 + 11 + 11^2 + 11^3 < 2010 \leq n$ and $1 + 13 + 13^2 + 13^3 > 2019 \geq n$. Therefore we must have $m = pq$ and $n = 1+q+p+pq = (1+q)(1+p)$. Because p is odd, n must be an even number. If $q = 2$, then n must be divisible by 3. In the given set only $2010 = (1+2)(1+669)$ and $2016 = (1+2)(1+671)$ satisfy these requirements. However neither 669 nor 671 are prime. If q is odd, then n must be divisible by 4. In the given set, only 2012 and 2016 are divisible by 4. None of the pairs of factors of 2012, namely $1 \cdot 2012$, $2 \cdot 1006$, $4 \cdot 503$, gives rise to primes p and q. This leaves $2016 = (1+3)(1+503)$, which is the only nice number in the given set.

Remark: Note that 2016 is nice in five ways. The other four ways are $(1+7)(1+251)$, $(1+11)(1+167)$, $(1+23)(1+83)$, and $(1+41)(1+47)$.

2002B 21. **(B)** Let A be the number of square feet in Andy's lawn. Then $A/2$ and $A/3$ are the areas of Beth's lawn and Carlos' lawn, respectively, in square feet. Let R be the rate, in square feet per minute, that Carlos' lawn mower cuts. Then Beth's mower and Andy's mower cut at rates of $2R$ and $3R$ square feet per minute, respectively. Thus,

$$\text{Andy takes } \frac{A}{3R} \text{ minutes to mow his lawn,}$$

$$\text{Beth takes } \frac{A/2}{2R} = \frac{A}{4R} \text{ minutes to mow hers,}$$

and

$$\text{Carlos takes } \frac{A/3}{R} = \frac{A}{3R} \text{ minutes to mow his.}$$

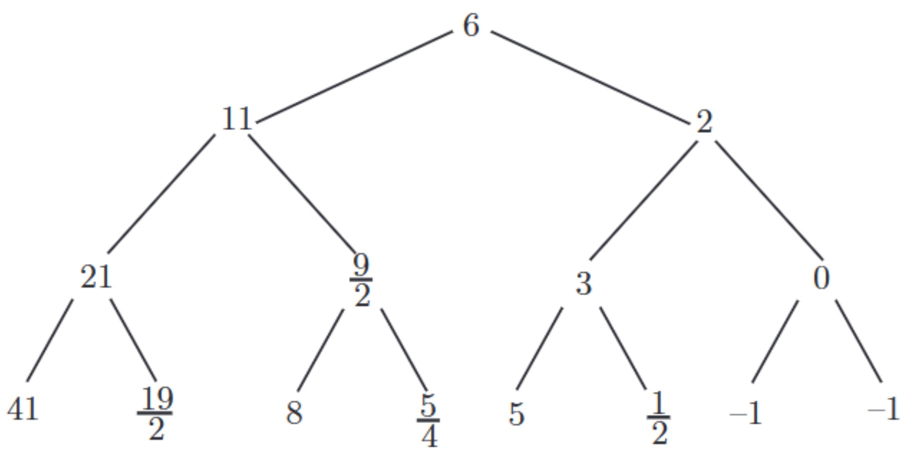
Since $\frac{A}{4R} < \frac{A}{3R}$, Beth will finish first.

2007A

23. **Answer (B):** Let x and y be, respectively, the larger and smaller of the integers. Then $96 = x^2 - y^2 = (x+y)(x-y)$. Because 96 is even, x and y are both even or are both odd. In either case $x+y$ and $x-y$ are both even. Hence there are four possibilities for $(x+y, x-y)$, which are $(48, 2)$, $(24, 4)$, $(16, 6)$, and $(12, 8)$. The four corresponding values of (x, y) are $(25, 23)$, $(14, 10)$, $(11, 5)$, and $(10, 2)$.

2008A

22. **Answer (D):** The tree diagram below gives all possible sequences of four terms. In the diagram, each left branch from a number corresponds to a head, and each right branch to a tail.



Because the coin is fair, each of the eight possible outcomes in the bottom row of the diagram is equally likely. Five of those numbers are integers, so the required probability is $\frac{5}{8}$.

2012A

22. **Answer (A):** The sum of the first k positive integers is $\frac{k(k+1)}{2}$. Therefore the sum of the first k even integers is

$$2+4+6+\cdots+2k = 2(1+2+3+\cdots+k) = 2\cdot\frac{k(k+1)}{2} = k(k+1).$$

The sum of the first k odd integers is

$$(1+2+3+\cdots+2k)-(2+4+6+\cdots+2k) = \frac{2k(2k+1)}{2} - k(k+1) = k^2.$$

The given conditions imply that $m^2 - 212 = n(n+1)$, which may be rewritten as $n^2+n+(212-m^2)=0$. The discriminant for n in this quadratic equation is $1-4(212-m^2) = 4m^2-847$, and this must be the square of an odd integer. Let $p^2 = 4m^2 - 847$, and rearrange this equation so that $(2m+p)(2m-p) = 847$.

The only factor pairs for 847 are $847\cdot 1$, $121\cdot 7$, and $77\cdot 11$. Equating these pairs to $2m+p$ and $2m-p$ yields $(m,p) = (212, 423)$, $(32, 57)$, and $(22, 33)$. Note that the corresponding values of n are found using $n = \frac{-1+p}{2}$, which yields 211, 28, and 16, respectively. The sum of the possible values of n is 255.

2003B

24. (E) Since the difference of the first two terms is $-2y$, the third and fourth terms of the sequence must be $x-3y$ and $x-5y$. Thus

$$x-3y = xy \quad \text{and} \quad x-5y = \frac{x}{y},$$

so $xy - 5y^2 = x$. Combining these equations we obtain

$$(x-3y)-5y^2 = x \quad \text{and, therefore,} \quad -3y-5y^2 = 0.$$

Since y cannot be 0, we have $y = -3/5$, and it follows that $x = -9/8$. The fifth term in the sequence is $x - 7y = 123/40$.

2004A 24. **(D)** Note that

$$a_{2^1} = a_2 = a_{2\cdot 1} = 1 \cdot a_1 = 2^0 \cdot 2^0 = 2^0,$$
$$a_{2^2} = a_4 = a_{2\cdot 2} = 2 \cdot a_2 = 2^1 \cdot 2^0 = 2^1,$$
$$a_{2^3} = a_8 = a_{2\cdot 4} = 4 \cdot a_4 = 2^2 \cdot 2^1 = 2^{1+2},$$
$$a_{2^4} = a_{16} = a_{2\cdot 8} = 8 \cdot a_8 = 2^3 \cdot 2^{1+2} = 2^{1+2+3},$$

and, in general, $a_{2^n} = 2^{1+2+\cdots+(n-1)}$. Because

$$1 + 2 + 3 + \cdots + (n-1) = \frac{1}{2}n(n-1),$$

we have $a_{2^{100}} = 2^{(100)(99)/2} = 2^{4950}$.

2015B 24. **Answer (D):** Note that for any natural number k, when Aaron reaches point $(k, -k)$, he will have just completed visiting all of the grid points within the square with vertices at $(k, -k)$, (k, k), $(-k, k)$, and $(-k, -k)$. Thus the point $(k, -k)$ is equal to $p_{(2k+1)^2-1}$. It follows that $p_{2024} = p_{(2\cdot 22+1)^2-1} = (22, -22)$. Because $2024 - 2015 = 9$, the point $p_{2015} = (22 - 9, -22) = (13, -22)$.

2016B 24. **Answer (D):** Let k be the common difference for the arithmetic sequence. If $b = c$ or $c = d$, then $k = bc - ab = cd - bc$ must be a multiple of 10, so $b = c = d$. However, the two-digit integers bc and cd are then equal, a contradiction. Therefore either (b, c, d) or $(b, c, d+10)$ is an increasing arithmetic sequence.

Case 1: (b, c, d) is an increasing arithmetic sequence. In this case the additions of k to ab and bc do not involve any carries, so (a, b, c) also forms an increasing arithmetic sequence, as does (a, b, c, d). Let $n = b - a$. If $n = 1$, the possible values of a are 1, 2, 3, 4, 5, and 6. If $n = 2$, the possible values of a are 1, 2, and 3. There are no possibilities with $n \geq 3$. Thus in this case there are 9 integers that have the required property: 1234, 2345, 3456, 4567, 5678, 6789, 1357, 2468, and 3579.

Case 2: $(b, c, d + 10)$ is an increasing arithmetic sequence. In this case the addition of k to bc involves a carry, so $(a, b, c - 1)$ forms a nondecreasing arithmetic sequence, as does $(b, c-1, (d+10)-2) = (b, c-1, d+8)$. Hence $(a, b, c - 1, d + 8)$ is a nondecreasing arithmetic sequence. Again letting $n = b - a$, note that $0 \leq c = d + (9 - n) \leq 9$ and $1 \leq a = d + (8 - 3n) \leq 9$. The only integers with the required properties are 8890 with $n = 0$; 5680 and 6791 with $n = 1$; 2470, 3581, and 4692 with $n = 2$; and 1482 and 2593 with $n = 3$. Thus in this case there are 8 integers that have the required property.

The total number of integers with the required property is $9 + 8 = 17$.

2007A 25. **Answer (D):** If $n \leq 2007$, then $S(n) \leq S(1999) = 28$. If $n \leq 28$, then $S(n) \leq S(28) = 10$. Therefore if n satisfies the required condition it must also satisfy

$$n \geq 2007 - 28 - 10 = 1969.$$

In addition, $n, S(n)$, and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n)$, and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \leq n \leq 1999$, we have $n + S(n) \geq 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

2010A

25. **Answer (B):** Let the sequence be $(a_1, a_2, \ldots, a_8)$. For $j > 1$, $a_{j-1} = a_j + m^2$ for some m such that $a_j < (m+1)^2 - m^2 = 2m+1$. To minimize the value of a_1, construct the sequence in reverse order and choose the smallest possible value of m for each j, $2 \le j \le 8$. The terms in reverse order are $a_8 = 0$, $a_7 = 1$, $a_6 = 1 + 1^2 = 2$, $a_5 = 2 + 1^2 = 3$, $a_4 = 3 + 2^2 = 7$, $a_3 = 7 + 4^2 = 23$, $a_2 = 23 + 12^2 = 167$, and $N = a_1 = 167 + 84^2 = 7223$, which has the unit digit 3.

2011B

25. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$\begin{aligned} 2BE &= BE + BD = BE + CE + BD + AD - (AF + CF) \\ &= a + c - b, \end{aligned}$$

that is, $BE = \frac{1}{2}(a+c-b)$. Similarly $AD = \frac{1}{2}(b+c-a)$ and $CF = \frac{1}{2}(a+b-c)$. In the given $\triangle ABC$, suppose that $AB = x+1$, $BC = x-1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$\begin{aligned} BE &= \frac{1}{2}((x-1)+(x+1)-x) = \frac{1}{2}x, \\ AD &= \frac{1}{2}(x+(x+1)-(x-1)) = \frac{1}{2}x+1, \text{ and} \\ CF &= \frac{1}{2}((x-1)+x-(x+1)) = \frac{1}{2}x-1. \end{aligned}$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y-1, y, y+1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of T_{n+1} equals $\frac{1}{2}(a+b+c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011 + 2012 + 2013) = \frac{1509}{2^{N-3}}.$$

The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

$$-1 + \frac{2012}{2^N} + \frac{2012}{2^N} \le 1 + \frac{2012}{2^N}.$$

Equivalently, $2012 \le 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.