17

QUADRATICS

2010A

21. **Answer (A):** Let the polynomial be $(x-r)(x-s)(x-t)$ with $0 < r \le s \le t$. Then $rst = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$, and $r + s + t = a$. If $t = 67$, then $rs = 30$, and $r + s$ is minimized when $r = 5$ and $s = 6$. In that case $a = 67 + 5 + 6 = 78$. If $t \ne 67$, then $a > t \ge 2 \cdot 67 = 134$, so the minimum value of a is 78.

2018A

21. **Answer (E):** Solving the second equation for x^2 gives $x^2 = y + a$, and substituting into the first equation gives $y^2 + y + (a - a^2) = 0$. The polynomial in y can be factored as $(y + (1 - a))(y + a)$, so the solutions are $y = a - 1$ and $y = -a$. (Alternatively, the solutions can be obtained using the quadratic formula.) The corresponding equations for x are $x^2 = 2a - 1$ and $x^2 = 0$. The second equation always has the solution $x = 0$, corresponding to the point of tangency at the vertex of the parabola $y = x^2 - a$. The first equation has 2 solutions if and only if $a > \frac{1}{2}$, corresponding to the 2 symmetric intersection points of the parabola with the circle. Thus the two curves intersect at 3 points if and only if $a > \frac{1}{2}$.

OR

Substituting the value for y from the second equation into the first equation yields

$$x^2 + \left(x^2 - a\right)^2 = a^2,$$

which is equivalent to

$$x^2\left(x^2 - (2a - 1)\right) = 0.$$

The first factor gives the solution $x = 0$, and the second factor gives 2 other solutions if $a > \frac{1}{2}$ and no other solutions if $a \le \frac{1}{2}$. Thus there are 3 solutions if and only if $a > \frac{1}{2}$.

2015A

23. **Answer (C):** The zeros of f are integers and their sum is a, so a is an integer. If r is an integer zero, then $r^2 - ar + 2a = 0$ or

$$a = \frac{r^2}{r-2} = r + 2 + \frac{4}{r-2}.$$

So $\frac{4}{r-2} = a - r - 2$ must be an integer, and the possible values of r are 6, 4, 3, 1, 0, and -2. The possible values of a are 9, 8, 0, and -1, all of which yield integer zeros of f, and their sum is 16.

OR

As above, a must be an integer. The function f has zeros at

$$x = \frac{a \pm \sqrt{a^2 - 8a}}{2}.$$

These values are integers only if $a^2 - 8a = w^2$ for some integer w. Solving for a in terms of w gives $a = 4 \pm \sqrt{16 + w^2}$, so $16 + w^2$ must be a perfect square. The only integer solutions for w are 0 and ± 3, from which it follows that the values of a are 0, 8, 9, and -1, all of which yield integer values of x. The requested sum is 16.

2000

24. **Answer (B):** Let $x = 9z$. Then $f(3z) = f(9z/3) = f(3z) = (9z)^2 + 9z + 1 = 7$. Simplifying and solving the equation for z yields $81z^2 + 9z - 6 = 0$, so $3(3z+1)(9z-2) = 0$. Thus $z = -1/3$ or $z = 2/9$. The sum of these values is $-1/9$.
Note. The answer can also be obtained by using the sum-of-roots formula on $81z^2 + 9z - 6 = 0$. The sum of the roots is $-9/81 = -1/9$.

2017A

24. **Answer (C):** Let q be the additional root of $f(x)$. Then

$$\begin{aligned} f(x) &= (x-q)(x^3+ax^2+x+10) \\ &= x^4+(a-q)x^3+(1-qa)x^2+(10-q)x-10q. \end{aligned}$$

Thus $100 = 10-q$, so $q = -90$ and $c = -10q = 900$. Also $1 = a-q = a+90$, so $a = -89$. It follows, using the factored form of f shown above, that $f(1) = \big(1-(-90)\big)\cdot(1-89+1+10) = 91\cdot(-77) = -7007$.

2010B

25. **Answer (B):** Because $1,3,5$, and 7 are roots of the polynomial $P(x)-a$, it follows that

$$P(x)-a = (x-1)(x-3)(x-5)(x-7)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. The previous identity must hold for $x = 2,4,6$, and 8, thus

$$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$$

Therefore $315 = \text{lcm}(15,9,105)$ divides a, that is a is an integer multiple of 315. Let $a = 315A$. Because $Q(2) = Q(6) = 42A$, it follows that $Q(x) - 42A = (x-2)(x-6)R(x)$ where $R(x)$ is a polynomial with integer coefficients. Because $Q(4) = -70A$ and $Q(8) = -6A$ it follows that $-112A = -4R(4)$ and $-48A = 12R(8)$, that is $R(4) = 28A$ and $R(8) = -4A$. Thus $R(x) = 28A + (x-4)(-6A+(x-8)T(x))$ where $T(x)$ is a polynomial with integer coefficients. Moreover, for any polynomial $T(x)$ and any integer A, the polynomial $P(x)$ constructed this way satisfies the required conditions. The required minimum is obtained when $A = 1$ and so $a = 315$.

2018B

25. **Answer (C):** Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Then $0 \le \{x\} < 1$. The given equation is equivalent to $x^2 = 10{,}000\{x\}$, that is,

$$\frac{x^2}{10{,}000} = \{x\}.$$

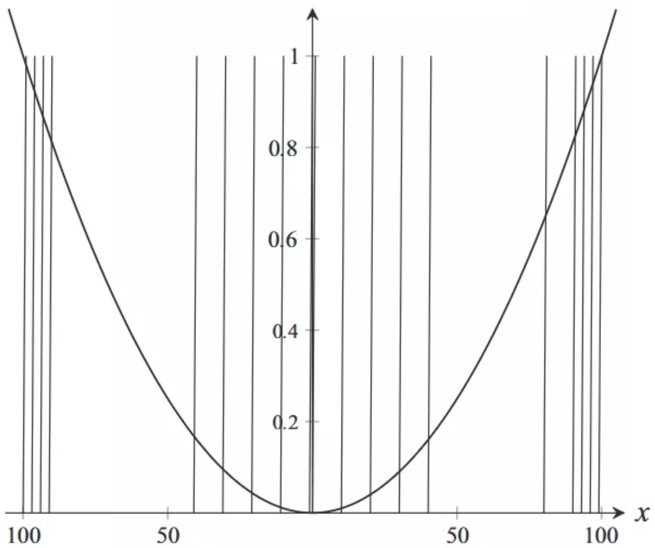
Therefore if x satisfies the equation, then

$$0 \le \frac{x^2}{10{,}000} < 1.$$

This implies that $x^2 < 10{,}000$, so $-100 < x < 100$. The figure shows a sketch of the graphs of

$$f(x) = \frac{x^2}{10{,}000} \qquad \text{and} \qquad g(x) = \{x\}$$

for $-100 < x < 100$ on the same coordinate axes. The graph of g consists of the 200 half-open line segments with slope 1 connecting the points $(k, 0)$ and $(k+1, 1)$ for $k = -100, -99, \ldots, 98, 99$. (The endpoints of these intervals that lie on the x-axis are part of the graph, but the endpoints with y-coordinate 1 are not.) It is clear that there is one intersection point for x lying in each of the intervals $[-100, -99)$, $[-99, -98)$, $[-98, -97)$, $\ldots$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $\ldots$, $[97, 98)$, $[98, 99)$ but no others. Thus the equation has 199 solutions.



OR

The solutions to the equation correspond to points of intersection of the graphs $y = 10000\lfloor x \rfloor$ and $y = 10000x - x^2$. There will be a point of intersection any time the parabola intersects the half-open horizontal segment from the point $(a, 10000a)$ to the point $(a+1, 10000a)$, where a is an integer. This occurs for every integer value of a for which

$$10000a - a^2 \le 10000a < 10000(a+1) - (a+1)^2.$$

This is equivalent to $(a+1)^2 < 10000$, which occurs if and only if $-101 < a < 99$. Thus points of intersection occur on the intervals $[a, a+1)$ for $a = -100, -99, -98, \ldots, -1, 0, 1, \ldots, 97, 98$, resulting in 199 points of intersection.